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# ECONOMIC DESIGN OF INDUSTRIAL PRODUCT INSPECTION SYSTEMS

A THESIS

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# FOREWORD

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# SUMMARY

The purpose of this study is to obtain a better understanding of the relationships existing between the design characteristics of an acceptance inspection system and the value of this system to the organization which employs it. This purpose is achieved through the development of a set of principles and a methodology which form the basis for decision in the design of acceptance inspection systems.

Acceptance inspection is defined as examination of product characteristics to determine their conformance to given specifications. On the basis of inspection results, a decision regarding the disposition of the product is made. Inspection may be carried out on each individual product unit or a sampling procedure may be used.

A survey of the literature of acceptance inspection is summarized by a detailed description of present methods of selecting acceptance inspection procedures, a classified bibliography on economic analysis of acceptance inspection, and an analysis of the shortcomings of current practice. The latter basically consist of the failure to fully utilize prior knowledge of the statistical properties of the process generating the product, the difficulty of relating the non-economic criteria now used in specifying inspection procedures to the economic well-being of the organization, the lack of recognition of the psychological influence of inspection on product quality, and the failure to treat inspection as a system of interrelated operations.

The theoretical framework for describing and analyzing the in-

spection system design problem is found in Abraham Wald's work on statistical decision theory. Decision theory, which describes the manner in which a decision maker jointly considers objectives, alternative courses of action, and various possible future circumstances in order to select the alternative which is optimal with respect to his measure of utility and principle of choice, is presented. Principles of choice are described for decision making under certainty, risk, and uncertainty. The framework is extended to encompass statistical decisions, wherein information from experimentation is utilized in making the decision. Statistical decision theory integrates economic and statistical considerations, and therefore its application to acceptance inspection requires development of the relevant economic and statistical principles.

Gains and losses influenced by acceptance inspection decisions are placed into three categories: inspection costs, acceptance losses, and rejection losses. A model is proposed for determining the after-tax annual cost of inspection. A method for computing the internal losses caused by passing a defective product is presented. It is shown that the losses associated with the rejection of a product depend upon the disposition of the material, and several possibilities are analyzed. The effect of income taxes upon these classes of losses is described.

It is concluded that monetary units represent the best available measure of utility and that the Bayes principle, perhaps constrained by specification of aspiration levels, is the logical criterion for acceptance inspection decisions. The noneconomic criteria used in designing commonly used inspection procedures are described, and the economic implications underlying the Dodge-Romig *Sampling Inspection Tables* and

*Military Standard 105D* are discussed.

To satisfy certain of the statistical requirements for the application of statistical decision theory to acceptance inspection, expressions were developed for the following quantities:

1. Conditional probability distribution of the sample outcome, given the lot quality.

2. Joint probability distribution of the sample outcome and the lot quality.

3. Marginal distribution of the sample outcome.

4. Conditional distribution of lot quality, given the sample outcome.

5. Probability of accepting a lot of a given quality.

6. Expected quality reaching the consumer under rectifying inspection and under nonrectifying inspection.

7. Average size of lots reaching the consumer.

8. Expected number of units inspected per lot.

These results are obtained for single sampling plans for attribute inspection for defectives, attribute inspection for defects, and variables inspection with known process standard deviation. Specific attention is given to the following prior distributions: hypergeometric, binomial, Polya, and mixed binomial distributions for defectives inspection; gamma and m-point discrete distributions for defects inspection; and normal distribution for variables inspection.

Several considerations, additional to those of a statistical or economic nature, are discussed. Methods for quantifying the effect of inspection errors by inspectors or instruments are given. A procedure is described for determining the economically optimum number of replicate measurements to make on an item. Brief qualitative statements are made about the influence of learning on inspection costs, the psychological effects of inspection, the possibility of competition between producer and consumer, and the organization for inspection. No attempt is made to formally analyze these latter considerations, because it is felt that no generally applicable theory could be developed within the scope of this research.

A conceptual model of the inspection function as a system of interrelated operations is presented, and the applicability of statistical decision theory as a method for the selection of inspection procedures is demonstrated. This approach to system optimization is described initially for a single-stage system. Two examples are given to illustrate the methodology: the first is an analysis of rectifying inspection and the second is an analysis of nonrectifying inspection. In both cases it is demonstrated that sampling inspection is never optimal if the prior distribution is binomial and the loss function is linear.

The methodology also is applied to a multistage system wherein an item has to be processed through a sequence of production operations and there is a maximum number of defects allowed a completed unit. The hypothesis that an economic gain could be realized from tightening the specifications for inprocess inspections is examined with respect to two problem environments: (1) a manufacturer producing to satisfy a fixed production quota, and (2) a manufacturer producing until he exhausts a fixed stock of raw material. Under specific assumptions regarding the disposition of rejected product, a dynamic programming model is formulated for each problem. A numerical example is solved for a three-stage process. Results indicate that artifically severe specification limits should be considered and that dynamic programming methods can be utilized to determine the most economic limits.

The use of a weighted defect total is suggested when various types of defects exist and they are not all of the same degree of severity. Specification limits would be established for the weighted total.

As a result of this investigation it is concluded that additional research would be beneficial in the development of procedures for determining costs and prior distributions, in quantifying the psychological effects of inspection, in determination of the optimal design of single-stage inspection systems for particular prior distributions and utility functions, in development and application of multistage optimization techniques to the design of acceptance inspection systems, and in the joint design of inspection systems for product acceptance and for process control.

The use of statistical decision theory in practice is made difficult by lack of knowledge of the prior distribution, uncertainty about estimates of losses, and complexity of computations required to obtain a solution. However, the theory does properly describe the decision problem, thereby giving the analyst an understanding of the nature of the problem which he is attempting to solve. In those cases where the prior distribution can be described adequately and losses can be estimated accurately, use of the theory permits choice of a system which best satisfies the objectives of the decision maker.

# CHAPTER I

# INTRODUCTION

## Purpose

The purpose of the research reported herein was to obtain a better understanding of the relationships existing between the design characteristics of acceptance inspection systems and the value of these systems to the organizations which employ them. Such knowledge would permit an objective approach to the problem of establishing inspection operations for determining the acceptability of product processed by an organization.

## Product Quality

Product acceptability is ascertained in relation to established quality standards. Since a precise definition of the word "quality" is required for proper interpretation of the scope of this research, a description of the method of determining these standards is in order.

### Quality Characteristics

Quality is a subjective measure evaluated by examination of the characteristics of a product which cause the user to attribute value to it. These characteristics will be called "quality characteristics." Different magnitudes of a quality characteristic will have different quality levels in the evaluation of the user. For a given function, the user may desire a certain level of quality and specify certain values for his quality characteristics. Because the realized value of a quality characteristic is subject to variation from a number of causes in the production process and because the user's quality function may be constant over a range of values for the characteristic in question, the user may state his requirement in the form of an interval of values. The limits on acceptable variation in a quality characteristic will be called "specifications." Quality characteristics and their specifications may be measurable quantities, such as dimensions, or they may be attribute quantities, such as broken or not broken.

It will be convenient to conceive of quality inherent in the design of the product and quality inherent in the conformance of a particular item to that design. Quality of design is affected by a change in the design specifications, while quality of conformance is a measure of how well the product adheres to given specifications. Acceptance inspection is concerned with the conformance of the product being inspected to specifications supplied by the designer.

Quality of Design

To the consumer, design quality is the net resultant of the combined functional, operative, and other attributes of the material or product being considered. This can be expressed symbolically as

$$Q = f(q_1, q_2, \ldots, q_n) ,$$

where Q is design quality and $q_1, q_2, \ldots, q_n$ are the functional or other attributes of the product.[1] Some q's can be measured quantitatively, while others cannot. Some can vary over a wide range without affecting

1. This discussion is based on one by Edwards (36).

Q, while others are critical. Often the market research group or product development group of a firm is faced with the problem of determining (usually subjectively) the nature of the function $f(\cdot)$ and desirable values for important q's, with the goal of producing a design having appeal to customers in the market. In other cases, a customer may specify values for the q's.

The attributes, $q_1, q_2, \ldots, q_n$, are functions of the physical characteristics of the product; i.e.,

$$q_i = g_i(p_1, p_2, \ldots, p_m)$$

where $p_1, p_2, \ldots, p_m$ are physical properties of the product. From the point of view of a production department, the p's are the quality characteristics of interest. Some of the p's will have little influence on any of the q's; others may affect more than one q. It is important that the product design group determine values for $p_i$, $i = 1,2,\ldots,m$, such that the required q's (and hence Q) will be inherent in the product. The selected p-values, along with allowable variation, may then be communicated to the production department in the form of blueprints and material specifications. The production department then has the responsibility of producing a product having the specified physical properties, $p_1,p_2,\ldots,p_m$.

For illustration, suppose the product is an automobile. An important functional characteristic (q) might be the rate of gasoline consumption, and physical properties (p), which might influence this q, would be weight, wheel diameter, engine efficiency, size of tires, and air pressure in tires.

Among the many interesting aspects of quality of design are those of an economic nature. Demand for the product usually will depend upon the design quality Q, resulting from chosen $q_1$, $q_2$, ..., $q_n$, as well as price, promotional effort, competition among buyers, general economic conditions, and availability of substitute products.[2] Thus, one may think of a value to the firm associated with each level of design quality. Further there is a cost associated with imparting the physical characteristics, $p_1$, $p_2$, ..., $p_m$, to the product. The objective of choosing Q to maximize the difference between the value and the cost may be illustrated with the following graph from Juran (73, p. 7).

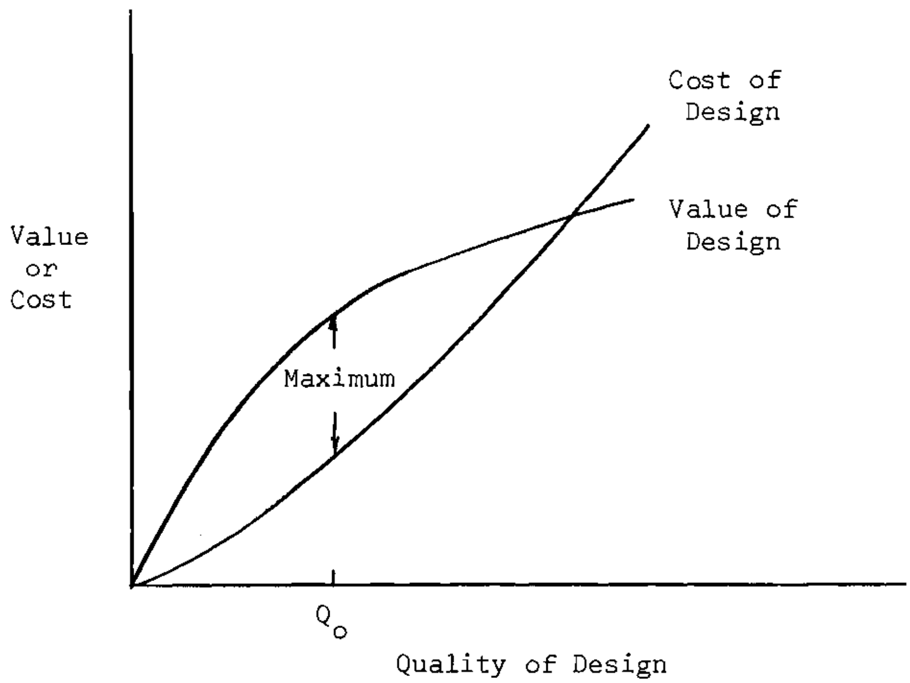


Figure 1. Optimal Level of Quality of Design

2. It may be that different consumers will have different quality functions. For example, products of the automotive industry cover a wide range of design.

The preceding discussion of quality inherent in design was entirely conceptual in nature. This is a consequence of a dearth of published research into problems of economic choice of design quality.[3]

## Quality of Conformance

The actual achieved quality may differ from the design quality for many reasons; e.g., misunderstood specifications, limitations on manufacturing capabilities, blunders in production, damage in storage, in transportation, or in installation. The degree to which a manufactured product conforms to its design specifications is a measure of conformance quality. It is this conformance or lack of conformance that is of concern to production management. A production department logically cannot be held responsible for shortcomings in design quality; however, it can be made accountable for deviations of actual product characteristics from design specifications.

Conformance quality is controlled through use of information obtained by inspection of the product at various stages of its manufacture. This information is used in two general ways: (1) to improve operation of the processes producing the product, in order to prevent the production of non-conforming items, and (2) to aid in making decisions regarding the disposition of product already processed. In the latter case, these decisions which "sentence" the product may be made at any stage of production. This includes every operation from purchased material ac-

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3. Two studies are those of Abbott (1) and Starr (104). Abbott gives a qualitative discussion of variation in design quality as a form of competition. Starr presents an elementary application of decision theory to the problem of choosing design specifications.

quisition to finished product shipping. Various sentences are imposed as a result of inspection; e.g., "accept as is," "rework," "scrap," "send back to the vendor," or "classify as a second."

It has been suggested (73, pp. 12-14, and 44, pp. 515-524) that there is some optimal level of effort in the control of conformance quality. Losses averted by improving conformance quality through programs of process control or acceptance inspection are offset by costs associated with the quality control program,[4] so that the most economic (in the sense of minimum total losses) level of conformance quality may be less than the production of a 100 per cent conforming product. This concept of optimality was illustrated by Juran (73, p. 8) and is reproduced in Figure 2.

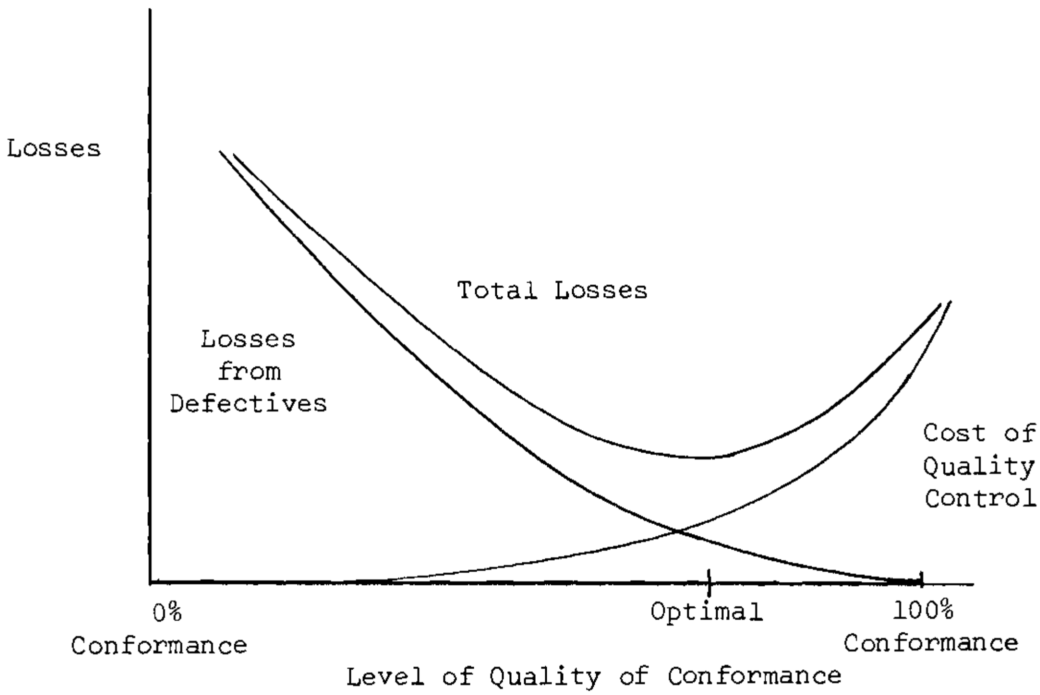


Figure 2. Illustration of Optimal Level of Conformance

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4. These losses will be discussed in detail in Chapter III.

In summary, it is assumed that a product has value not only because of its design characteristics, but also because of its conformance to the specifications on these characteristics. Hereafter, unless stated otherwise, use of the word "quality" will refer to quality inherent in the conformance to a given design.

## The Quality Control Function

In order to maintain a desirable level of product conformance to specifications, most organizations have quality control programs. That is, elements within the organization have definite responsibilities regarding the acquisition, distribution, and analysis of information relevant to quality and the making of various decisions about product or process on the basis of this information. Purchased material, work in progress, and finished goods may be subject to formal controls. Techniques for control usually are classified into two major areas: process control and product control.

Process control is a preventive endeavor, in which efforts are made to regulate the production processes to obtain product conformance to specifications. Control charts, first piece inspection, patrol inspection, and process capability studies are typical of process control techniques.

Product control is a corrective activity, in which efforts are made to detect and rectify poor quality in material already processed. Typically, this involves the inspection of material to determine its acceptability with regard to specifications.

In both types of control, data for decision making are obtained primarily from inspection of the product. While the same information may be used for both product and process control, the considerations affecting decisions in the two areas differ substantially. This point is discussed in a number of texts; for example, Grant (44, Chapter 1) and Juran (73, Chapters 4 and 5).

## Inspection

Inspection of a product involves the examination of its characteristics and a comparison with standards established for them. Juran (73, p. 31) describes functions of an inspector as follows:

1. Interpretation of specifications.
2. Measurement of the product.
3. Judgment as to conformance by comparison of (2) with (1).
4. Disposition of product inspected.
5. Recording of data obtained.

### Justification for Inspection

The particular types of inspection operations which exist in a manufacturing process are established by necessity, policy, and economics.

Production management may consider inspection a necessity if the safety of men or equipment is jeopardized by a defective[5] product, if the contract with the customer requires it, or if it is fundamental to manufacturing (e.g., needed to guide setup operations or to classify the product).

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5. A defective product is one which fails to conform to specifications.

Inspection decisions often are made on the basis of policies or traditions, which stem from sources such as pride in the company's record of quality or preconceived beliefs about the "best" procedures for inspection. In such cases, explicit economic justification of inspection decisions is not attempted.

Economic justification of inspection decisions is usually of a subjective nature. Present practice seems to involve a qualitative evaluation of the economic consequences in order to make a given inspection decision. On the basis of this type of decision process, certain principles of inspection have been advanced.[6] By the application of these principles, one is supposed to be able to make decisions which are in accord with the economic objectives of the firm. In many cases, the economic factors may be amenable to explicit analytical methods and the inspection system designed by use of a formal procedure. Examples of the latter approach are rare.

Benefits from Inspection

Detection of Defective Product. Inspection may reveal a defective product, thereby permitting certain losses to be averted. The expenditure of resources in further processing defectives is avoided. Service and guarantee costs, as well as customer good will losses, are reduced. Purchased material may be returned and an adjustment received from the vendor, or reworkable product may be identified before future operations make recovery impossible. Costs of assembly may be lowered, or damage

6. Some of the principles which dictate the selection of acceptance sampling plans will be discussed in Chapter II.

to equipment may be avoided.

Information. Inspection yields information which may be used to improve quality generated by the process. Improper setups and undesirable changes in process parameters may be discovered. Vendor quality can be rated and vendors can be supplied with information to help them improve their quality. Process capabilities can be evaluated and information used in product redesign. Additionally, data on quality performance is necessary for cost analyses, production planning and control functions, facilities planning, and (as will be shown later) planning and improvement of inspection activities.

Prevention. The mere presence of an inspection operation may result in an improvement in quality performance at prior production operations or in the quality of purchased material. Inspection, when combined with penalties for poor quality and perhaps rewards for exceptional quality, can have a beneficial preventive effect.

Customer Relations. Knowledge that a producer has an adequate inspection system in operation may create confidence in the buyer's mind. The result may be new business or improved relations with existing customers.

## Classification of Inspection

The following are some of the many ways the inspection function has been classified:

Control Sampling. This involves the inspection of a sample of the output of a process for the purpose of determining if a change in process parameters has occurred or if the trend within the process is such that there is danger of producing a defective product.

First-Piece Inspection. The initial output from a process is inspected to judge the adequacy of the setup.

Audit Inspection. This is sampling inspection carried out occasionally to evaluate the quality performance of the process. This may be used to check the accuracy of routine inspection work.

Detail Inspection. This type of inspection is also called "100% inspection," "screening," "item inspection," or "sorting." Its purpose is to classify the product into categories by the inspection of every item.

Acceptance Inspection. This is inspection carried out to classify the product as acceptable (non-defective, effective, good) or unacceptable (defective). It may consist of either detail inspection or sampling inspection. If carried out on material purchased from another company, it is called "vendor (or receiving) inspection." If done between operations in the same company, it is called "in-process inspection." If done by the producer before shipment of finished goods to the buyer, it is referred to as "final inspection."

Acceptance Sampling. This is acceptance inspection where the decision regarding the acceptability of an amount of product is made on the basis of the quality of a sample from that product.

Lot-by-Lot Inspection. If the product is submitted for acceptance inspection in lots or batches, the inspection process is called lot-by-lot inspection.

Continuous Inspection. If the product is not formed into lots but rather is submitted for inspection an item at a time, the inspection process is referred to as continuous inspection.

Rectifying Inspection. This term most often is applied to detail inspection of the remainder of a lot when sampling inspection has indicated the lot is of poor quality. It also refers to detail inspection of a continuous output. Defectives discovered in this inspection usually are assumed to be repaired or replaced with effectives, so that the number of items (yield) from the process is not reduced.

Variable Inspection. In this type of inspection, quality characteristics are measured and decisions are made on the basis of the magnitude of these variables.

Attributes Inspection. In this type of inspection, the inspector notes only the number of ways in which an item fails to conform to specification. He records only whether or not the item was defective, or, in some instances, the number and type of defects in the item.

## Acceptance Inspection

This present investigation involves a study of acceptance inspection; therefore, some further discussion of this type of inspection is presented in the following sections. (Unless stated otherwise, the word "inspection" will refer to acceptance inspection.)

### Concept of an Inspection System

A manufacturing process may be conceived as a pattern of interrelated operations for the acquisition, production, and distribution of material.[7] It will be convenient to think of handling, transportation,

7. The terms "production" and "distribution" are not used in the broad economic sense, but rather they refer to the restricted acts of manufacturing within a plant and the shipping and warehousing which follow production.

inspection and storage as operations, as well as processes which are designed to change the properties of material. It is possible for product to be damaged in handling, transportation, or inspection and the quality of material could deteriorate in storage.

There will be a particular routing of material associated with the production of a given product. Opportunity for inspection exists before and after every other type of operation. In planning the location, physical characteristics, and operating procedures for inspection operations, one is designing a system for acceptance decisions. (More properly, inspection is a subsystem of the production system.) The nature of this system is determined by a number of decisions, to be described in the next section.

## Decision Problems

The decision problems listed below indicate the scope and complexity of inspection system design.

1. The quality characteristics to be inspected must be determined.

2. The locations in the production process for the inspection of each characteristic must be specified.

3. A choice between attribute or variables inspection may be necessary.

4. Grouping of quality characteristics for an attribute inspection must be decided upon.

5. A decision between lot-by-lot inspection and continuous inspection may be required.

6. A choice between sampling inspection and detail inspection

is necessary when inspection is not destructive.

7. The size of lots and the method of lot formation are decision variables if lot-by-lot inspection is to be used.

8. The parameters of sampling plans must be determined if acceptance sampling is utilized. This involves selection of the type of sampling scheme, sample size, and acceptance criteria.

9. The method by which the sample is extracted from the lot must be specified.

10. In acceptance sampling, a decision on allowing curtailed inspection must be made. Under this procedure, inspection is terminated as soon as the rejection number is reached.

11. Specifications to be used in an in-process inspection of a characteristic must be determined. These may differ from design specifications.

12. The disposition of rejected material must be decided upon.

13. The level of investment in inspection facilities and equipment may need to be specified.

14. Inspection equipment must be selected.

15. Inspection methods must be determined.

16. Number and assignment of inspectors must be planned.

17. Procedures for the pay of inspectors must be established.

18. The sequence in which multiple characteristics are inspected at an inspection operation must be determined.

19. The rate of inspection may need determination (as in the case of a conveyor-paced inspection station).

20. Accounting procedures for the allocation of inspection costs

must be planned and agreed to by the departments involved.

21. Procedures must be specified to allow acquisition and feedback of data for improvement of inspection operations.

22. Decisions may be needed regarding the extent of cooperation with the seller in vendor inspection and the buyer in final inspection.

Complicating Factors

Rational and explicit decision making is made difficult by a number of complications. Some of the more important are contained in the following list.

1. Inspection is not completely reliable. Items may be misclassified because of both human error and instrument error. Accuracy may be a function of many controllable factors; for example, inspection rate, lot size, sample size, quality of material inspected, inspection methods.

2. Costs needed for explicit economic analysis generally are not found in accounting records. Inspection costs may be related to the quality of the product being inspected. Inspection may create defectives, even in non-destructive testing. Losses resulting from shipping poor quality material to customers are difficult to estimate.

3. Psychological effects of inspection are not well known. Suppliers, customers, workers at prior operations, and workers at future operations may react to an inspection procedure.

4. Theories and results from statistics and probability are needed to describe quality variations and to predict performance under given inspection procedures. These subjects are complicated, and results for acceptance inspection are incomplete and are scattered in the literature.

5. The decisions listed in the previous section cannot be treated independently. In general, there is difficulty in expressing the interdependencies objectively. There is no standard measure of effectiveness for inspection decisions, so there exists no basis for determining the most sensitive decision variables.

Need for Research

The difficulties pointed out in the previous section result from a lack of fundamental understanding of the factors which influence inspection decisions. Most current research is concentrated in the area of acceptance sampling, with heavy emphasis on statistical characteristics and little or no treatment of other important considerations. There is a need for a frame of reference within which research can proceed and a need for better understanding of the economic principles involved. As inspection comes to be treated as a system rather than as an isolated operation, these needs become even more pronounced.

## Scope and Limitations

The present study takes a broad approach to formal analysis of inspection problems from the system viewpoint. Economic, statistical, psychological, and mathematical theories have been examined to extract useful principles, with the hope that the results will provide a foundation upon which improved solutions to specific inspection problems may be based.

The investigation reported herein is limited to product inspection problems created by the necessity for producing a product to conform to specifications established for its physical characteristics.

The particular form of inspection considered is that type referred to as "acceptance inspection." This inspection is for the purpose of deciding whether or not material should be allowed to proceed to the next stage in the acquisition-production-distribution system. Thus, inspection for process control purposes is not explicitly considered, although it is recognized that information from acceptance inspection operations may be of value in controlling the manufacturing process.

The environment studied consists of a series of operations typical of a manufacturing organization. These operations may be concerned with receipt of purchased material, processing of material through production operations, storage of material, transportation of material, inspection of material, and possible installation of the product at a customer's facility. It is assumed that all of the operations which affect product quality are sufficiently stable to allow the existence of a "process curve."[8]

Both variable and attribute inspection are considered, although primary emphasis is on attribute procedures. In general, it is assumed that material is presented for inspection in lots and that acceptance decisions are made with reference to the entire lot rather than to individual items. However, attention is given to detail inspection, especially in comparison with sampling inspection.

The research is a theoretical analysis of the effect of decisions regarding inspection activities on the economic well-being of the organ-

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8. The process curve is explained in detail in Chapter IV. Briefly, it is the probability distribution of the parameters of a process.

ization. No attempt was made to analyze a particular organization's production process.

## Objectives

The general objective of this research is the development of a set of principles and a methodology by which problems attendant to the design of inspection systems may be solved.

The specific objectives are the following:

1. To provide a summary and a critical review of techniques now in use or proposed for use in designing acceptance inspection systems.

2. To provide an exposition of the economic principles relevant to acceptance inspection and to point out the economic implications of commonly used procedures.

3. To consolidate statistical principles and results useful in the analysis of acceptance inspection systems.

4. To demonstrate other considerations in inspection system design: psychological factors, organization relationships, and mathematical techniques for system optimization.

5. To present a conceptual model of an acceptance inspection system and to analyze certain decision problems as suboptimizations.

6. To apply the principles and techniques exposed in accomplishing the above-mentioned objectives to a specific problem--that of selecting inspection procedures for a multistage production process, where the specifications are in terms of defects per unit.

7. To define areas where additional research would be of benefit to those responsible for establishing inspection procedures.

# CHAPTER II

# DEVELOPMENTS IN THE ANALYSIS AND DESIGN OF ACCEPTANCE INSPECTION

## General

A summary of literature reporting formal analysis of acceptance inspection systems is presented in this chapter. Most research has dealt with acceptance sampling procedures, applied at a single inspection station. Only recently has any work appeared in the literature in which inspection is analyzed as a system of interrelated operations. Earlier sections of this chapter summarize conventional inspection procedures, while the last sections present more recent work.

## Analysis of Single-Stage Inspection Systems Using Noneconomic Criteria

### Sampling Inspection Plans

A sampling inspection plan must specify the size of the sample, the information to be gathered from the sample, and decision rules which sentence product from which the sample was taken. Formal procedures for analyzing and specifying acceptance sampling plans stem from the early work of Dodge and Romig (32).

Initially, most research workers visualized only a single inspection operation for application of sampling plans. Also economic characteristics of the plans were not considered explicitly prior to 1945. However, during this period some important and useful procedures were developed and now form the basis for current practice. A brief summary of the principal results from noneconomic analysis of a single-stage inspection system will be required to appreciate later efforts toward inclusion of economic criteria and extension of analysis to multistage processes.

Attribute Sampling Schemes for Lot-by-Lot Inspection

General Description. The basic instructions for lot-by-lot attribute inspection are:[1]

(1) Take a sample of size $n_1$ items at random from the lot. Let $d_1$ denote the number of defectives in this first sample. The decision rule is:

(a) Accept the lot if $d_1 \le c_1$.

(b) Reject the lot if $d_1 \ge r_1$.

(c) Take a second sample of $n_2$ items if $c_1 < d_1 < r_1$.

(2) If the second sample is required,

(a) Accept the lot if $d_1 + d_2 \le c_2$.

(b) Reject the lot if $d_1 + d_2 \ge r_2$.

(c) Take a third sample if $c_1 < d_1 + d_2 < r_2$.

$$\vdots$$

In general, if the kth sample is required,

(a) Accept if $\sum_{i=1}^{k} d_i \le c_k$.

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1. The parameter pair $(c_k, r_k)$ are non-negative integers such that $c_k < r_k$, and $d_k$ is usually the number of defective items found in the kth sample. In some situations, $d_k$ may be the number of defects in the kth sample.

(b) Reject if $\sum_{i=1}^{k} d_i \geq r_k$.

(c) Take another sample of size $n_{k+1}$ in case

$$c_k < \sum_{i=1}^{k} d_i < r_k.$$

Particular forms of this scheme have been given special names.

Single Sample Plan. If $r_1 = c_1 + 1$, the plan is called a single sample plan because a terminal decision is forced after the first sample. Such a plan has the advantage of administrative simplicity and a fixed sample size per lot; however, this sample size may exceed the average sample number (ASN) possible under some other sampling scheme.[2]

Double Sample Plan. When $r_1 > c_1 + 1$ and $r_2 = c_2 + 1$, the plan is called a double sample plan. This type of plan may be preferred over a single sample plan, because it usually is possible to select a double sample plan such that for certain values of lot quality (as measured by the lot fraction defective) the ASN is less than that of a "matched"[3] single sample plan. This means a lower inspection cost for the same degree of protection. Also, in giving the lot a second chance when the first sample's quality is neither extremely good nor extremely poor, there may be some beneficial psychological effect on the producer of the lot. A disadvantage of double sampling procedures is the variation in inspection work from one lot to the next, since the actual number of

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2. The ASN is the expected number of units inspected per lot, not considering rectifying inspection of the remainder of rejected lots.

3. Matched sampling plans have approximately the same chance of accepting a lot of a given fraction defective; that is, they have the same operating characteristic (OC-) curve.

units inspected per lot is a random variable having a probability distribution which depends upon the size and fraction defective of the lot.

Item-by-Item Sequential Sampling Plan. When $n_k = 1$, for $k = 1,2,\ldots$, and the acceptance and rejection numbers are given by $c_k = a + bk$ and $r_k = a' + bk$, where $a' > a$ and $b > 0$, the plan is called an item-by-item sequential sampling plan (or just sequential sampling plan). The average sample number is small compared to matched single sample plans. However, the plan is tedious to administer because a decision must be made after each unit is inspected to determine whether the lot may be accepted or rejected, or whether another unit must be inspected. Also, the sample size varies from one lot to another.

Group Sequential Sampling Plan. If $n_k = n$, for $k = 1,2,\ldots$, and the acceptance and rejection numbers are given by $c_k = a + bnk$ and $r_k = a' + bnk$, where $a' > a$ and $b > 0$, the plan is named a group sequential sampling plan. It is a modification of the item-by-item sequential scheme to reduce the number of decision points at the sacrifice of an increase in the average sample size.

Multiple Sample Plan. A group sequential sampling plan that has been truncated by defining $r_k = c_k + 1$, for some $k > 2$ is called a multiple sample plan. (Note that for $k = 1$, a single sample plan would result and for $k = 2$, a double sample plan is obtained.) Truncation of the plan insures that a terminal decision will be made before the cumulative sample size grows beyond that required to make a reliable decision on the lot.

Serial Sampling Acceptance Schemes. Recent papers by Hill, *et al.* (65) and Cox (26) have contained discussions of deferred sentencing rules, wherein the decision on acceptance or rejection of a lot is based on data not only from that lot but also from lots immediately preceding or immediately following the one in question. This type of scheme would be used only when all lots are generated by the same process, so that there would be some reason to expect correlation between the quality of lots produced at about the same time.

Published Tables of Attribute Plans. Two sets of tables are commonly used as sources of lot-by-lot attribute plans. Tables prepared by Dodge and Romig (33) are designed for use when rejected lots are to be detail inspected, and Military Standard 105D (115) is to be used in cases where rejection procedures do not call for rectification of the remainder lot. In practice the two often are used without regard to the assumptions about the disposition of rejected lots.

The Dodge-Romig tables contain single and double sample plans which are designed to have the following characteristics for a given process average fraction defective, $\bar{p}$ :

1. Average total inspection per lot is to be a minimum when the process fraction defective is $\bar{p}$.

2. Either the average outgoing quality limit (AOQL)[4] is to have a specified value, or a specified level of poor lot quality, called the lot tolerance per cent defective (LTPD), is to have only probability 0.10 of being accepted with the plan.

Under the AOQL procedure, inspection costs are minimized when the

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4. The AOQL is the maximum value of the average outgoing quality, which depends upon incoming lot quality and the sampling plan. It is expressed as a fraction defective.

process is at its usual average level; however, the consumer is protected if quality should change--the AOQL being the worst possible quality he could receive over the long run.

The LTPD is similar except that the consumer is protected through specification of the smallest lot fraction defective which has at most one chance in ten of being accepted.

The economic factors of inspection cost and losses associated with acceptance of poor quality obviously were considered, but only subjectively. Use of these tables does not require estimates of any cost factors. For a given choice of AOQL or LTPD, the plan is determined by the lot size and the estimated process average fraction defective under normal conditions. The sample size increases with lot size and the process average. However, so does the acceptance number. Hald (54) has derived explicit asymptotic formulas for sample size as a function of lot size and for acceptance number as a function of sample size for the Dodge-Romig LTPD and AOQL single sample inspection plans.

Military Standard 105D is a collection of sampling plans, utilized for vendor inspection by agencies of the Department of Defense. These plans stem primarily from work done by the Statistical Research Group (105) at Columbia University during World War II. Entry into the tables requires specification of three quantities: lot size, acceptable quality level (AQL), and inspection level. The acceptable quality level is defined as a per cent defective that is considered acceptable as a process average. There are three general inspection levels and four special inspection levels, the latter being used when small samples are necessary. In general, lot size and inspection level completely determine sample size and, for that sample size, AQL determines acceptance number. Single, double, and multiple sample procedures are included and are matched by code letter[5] and AQL. The procedure has provision for "tightened" and "reduced" inspection. These are plans having more strict and less strict operating characteristic curves, respectively, than does the corresponding "normal" inspection plan. Under MIL-STD-105D, information from inspection of prior lots is used to determine whether normal, tightened, or reduced inspection is to be used for future lots.

There are other lesser-known sources for attribute sampling plans. Hamaker, *et al.* (56, 57, 58) have developed a system called the Phillips Standard Sampling System, which has had considerable use in Europe but only limited application in the United States. These tables are based upon a specified value for the "point of control" and the slope of the operating characteristic curve at that point. The point of control ($p_{.50}$) is the value of lot quality which has probability 0.50 of being accepted. The point of control could be thought of as an indifference quality, in that the inspecting organization is willing to assign equal probabilities of acceptance and rejection to lots of this quality. The relative slope of the operating characteristic curve at $p_{.50}$ is made a function of both the lot size and the point of control, although the relationship is not based on precise rules. Single sampling is used for lot sizes up to 1000 units and double sampling for lots of larger sizes. In double sample plans, $c_2 = 5\ c_1$, for $c_1 > 0$.

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5. The code letter is determined by lot size and inspection level.

Tables for constructing single sample plans having a specified operating characteristic curve were given by Cameron (20). The user must specify a level of "good" quality (called the producer's risk point), its corresponding probability of acceptance (called the producer's risk), a level of "poor" quality (called the consumer's risk point), and its associated probability of acceptance (called the consumer's risk). Thus, the user selects two points on the operating characteristic curve and this is sufficient to determine n and c. Cameron assumes the lot size to be large, so that the OC-curve may be computed from the Poisson distribution.

Golub (43) has presented charts for determining single sample plans when the sample size and consumer and producer risk points are specified, and the sum of the two risks is minimized.

Variables Sampling Schemes for Lot-by-Lot Inspection

General Description. When the product's acceptability is dependent upon one measurable characteristic, important advantages may result from using an acceptance plan based on statistics computed from measurements of a sample from the lot. Because the use of the measured value of the quality characteristic gives rise to statistics with a greater information content than those obtained by simply classifying each item as defective or effective, fewer units need be inspected in variables sampling to yield the same degree of protection afforded by an attribute plan. However, the unit cost of variables inspection may exceed that of attribute inspection. Also, many commonly used variables schemes require the assumption of a normally distributed quality characteristic.

Variables plans are of two general types: those designed on the basis of lot acceptability being a function of the lot mean, the lot standard deviation, or both, and those designed on the basis of lot acceptability being a function of the lot fraction defective.

Duncan (34) discusses the design of both types of variables plans. Essentially, plans based on the lot mean as the criterion for desirability of the material test an hypothesis about the mean of the lot, using a normal test if the lot standard deviation is known and a t-test otherwise. If the within-lot variability changes from lot-to-lot, taking on undesirable values in some cases, a chi-square test may be employed to test an hypothesis about the lot standard deviation. Tests mentioned require the assumption of normality, especially for small samples.

Plans based on the lot fraction defective as the criterion for desirability of the material either convert AQL's and LTPD's into values of lot means (assuming normality and known standard deviation) and proceed as a test of hypothesis about a mean, or they use sample measurements to estimate the fraction defective of the lot.

When the lot variability is unknown, the sample standard deviation is used as an estimate. For small samples, the sample range may replace the sample standard deviation with little loss in efficiency. However, if the sample size is large, say greater than 12, the range becomes inefficient. To combat this, the sample is divided into subsamples (usually of five units), the range of each subsample is computed, and the sample ranges are averaged. This average range, together with the sample mean, is used to test the hypothesis about the lot mean.

<u>Military Standard 414</u>. MIL-STD-414 (112) contains plans for the following situations:

1. The quality characteristic is normally distributed, with a known standard deviation. The sample average is used as an estimate of the lot mean to allow estimation of the lot fraction defective. If this estimated fraction defective exceeds the maximum allowable value specified in the plan, the lot is reject.

2. The quality characteristic is normally distributed, but the standard deviation is not known. The plan specifies the sample size and the maximum allowable estimate of the lot fraction defective. The fraction defective is estimated by use of a statistic based on the sample mean and sample standard deviation (standard deviation method) or a statistic based on the sample mean and the average of ranges of subgroups of size five formed from the sample data (average range method).

The lot size, acceptable quality level, inspection level (there are five levels in MIL-STD-414), and type of estimation procedure determine the plan. The AQL is defined as a nominal value expressed in terms of per cent defective specified for a single quality characteristic. The quality characteristic may have single or double specification limits which enter into the estimation of the lot fraction defective.

MIL-STD-414 provides procedures for estimating the process average and criteria for tightened and reduced inspection based on the inspection results of preceding lots.

This standard contains a special procedure for mixed variables-attributes sampling plans for use under certain conditions.[6] The vari-

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6. If the producer has screened his product before submitting the lot for inspection, lot quality will be truncated near the specifications. Mixed plans may be applied in this case.

ables plan is applied in the usual manner and if the lot is accepted no further sampling is done. If the lot is not accepted, the corresponding single sample attribute plan from MIL-STD-105D, using tightened inspection, is applied. The lot is accepted if it passes the attributes test. Thus, it must fail both tests to be rejected.

Bowker-Goode Sampling Inspection Tables. An earlier set of tables was provided by Bowker and Goode (14). From these tables, factors are available to allow construction of known-sigma plans based on the sample mean and applicable to both single and double specification limits. Also, there are factors for unknown-sigma plans for single specification limits. OC-curves are provided and matched single sample attribute plans are given.

The user may select a plan in one of two ways: he may select a satisfactory OC-curve (given as a function of the lot fraction defective) and employ the corresponding plan, or he may use the sample size code letter and AQL from the Statistical Research Group tables (105) to look up a Bowker-Goode plan. The acceptance criteria are in terms of the sample mean and standard deviation.

To a large extent these tables have been superceded by MIL-STD-414.

Shainin Lot Plot Method. Shainin (98) devised an acceptance scheme based on the frequency distribution of a sample of size 50. Although the plan received initial attention because of its uniqueness, its use does not appear extensive.

Variables Plans Where Lot Mean is Criterion for Lot Desirability. As stated earlier, situations of this type pose an hypothesis about the lot mean; therefore, classical statistical methods for testing hypotheses about the mean of a normal random variable can be utilized. Duncan (34) describes the usual methods and additionally presents a sequential plan based on the cumulative sum of sample measurements. Necessary equations for derivation of acceptance and rejection criteria are given.

Life Testing. Recently great emphasis has been placed upon the development of test procedures for determining the life characteristics of product subject to failure.[7] No doubt the increasing importance of reliability of systems of equipment has stimulated this activity. The problem is to collect data on time-to-failure of items either under field conditions or simulated field conditions or in accelerated tests, and then to use this data to estimate the distribution function of unit life. Most tests of this nature appear to be for the purpose of determining acceptability of product design or production process, although they may be utilized in acceptance inspection of manufactured product.

Many of these acceptance sampling plans assume an exponential distribution of life and use an estimate of the mean life as the basis for acceptance or rejection of the lot. Similar procedures have been developed for the Weibull distribution (e.g., U. S. Government's Quality Control and Reliability Technical Report TR 6), and the normal, gamma, and log-normal distributions appear in the literature (80).

A set of sampling plans, H 108, has been published by the Department of Defense (114). H 108 is based on an exponential life distribution and provides for the three cases where life tests are terminated

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7. The paper by Epstein and Sobel (38) is a basic work in this field.

upon the occurrence of a preassigned number of failures, where life tests are ended at a preassigned time, and where a sequential stopping rule is utilized. The user selects a plan with reference to the desired average life and the OC-curves given. The plan specifies the size of the sample, the stopping rule, and the acceptance criterion. H 108 provides for life testing both with replacement and without replacement of failed units.

Bulk Sampling. When product is in bulk form (for example, coal, cotton, gasoline), acceptance sampling usually involves estimation of the mean value of a characteristic of the product through analyses of material sampled. The acceptance decision is then based on a comparison of product specification and estimated mean, considering possible sampling error. Because statistical considerations are not as clearly defined as in sampling from a lot of separable items, this case has received special attention in the literature (35). There seem to be no general sampling procedures for bulk inspection, although certain industries have standards prescribing sampling methods and test procedures for their products. These standards appear to be arbitrary.

## Sampling Plans for Continuous Inspection

In many instances it is more convenient not to form production into lots, but rather to inspect the "continuous" stream of items, using plans referred to as continuous sampling plans. Dodge (30) proposed a plan of this nature as early as 1943. Since that time a multitude of plans have been advanced--some variations of the Dodge plan, others patterned after the sequential plans of Wald (121). These plans are a continuous form of sampling inspection with rectification, because they involve rectifying inspection until the process has shown itself consistently "good," at which time sampling inspection is introduced. If poor performance is observed under sampling inspection, rectifying inspection must be reinstated. In some cases provision is made for shutting down the process, thereby integrating the functions of process control and product control.

The Department of Defense has published H 106, a handbook of attribute continuous sampling plans (113). It is a modification of the Lieberman-Solomon multilevel sampling system (15, pp. 537-541). The user chooses an AOQL value as a part of the procedure of entering the table, so that he protects himself against receiving long-run quality in excess of this number. He also must select a sampling fraction and the number of inspection levels, which means that he has some control over inspection costs.

Another government publication, H 107, presents tables and operating procedures for several variations of the single-level plan developed by Dodge.

Control Charts

Often mentioned in the literature (44, pp. 451-452) are possibilities for the use of control charts for lot acceptance purposes. Since a control chart is merely a graphical device which facilitates the test of an hypothesis on a repeating basis, such statements are not surprising. The advantage of the control chart lies in the time variable used on the horizontal axis. This allows the user to compare present results with past patterns, thereby integrating process control and acceptance sampling.

Simon (101) suggests the use of a control chart to identify "grand lots," which may be sentenced on the basis of all data from the grand lot. A grand lot is the aggregate of all lots produced under essentially the same operating conditions. When Simon's method is used, the acceptance or rejection of individual lots must be postponed until the grand lot has been identified. This is a form of deferred sentencing.

Complex Sampling Schemes

In the preceding plans, the sample was assumed to have been selected at random from the entire lot--a procedure called simple random sampling. There are numerous situations in industrial acceptance inspection where this procedure is not desirable. For example, suppose that a firm purchases flashlight batteries which are received in crates of ten cartons each, the cartons each containing 24 boxes of five batteries. With simple random sampling the inspector might be forced to take each item from a different crate. An alternative might be to select a small random sample of crates, a sample of cartons within each crate selected, a sample of boxes within each selected carton, and finally a sample of items within each selected box. This would be an example of multistage sampling.

Techniques of multistage sampling, cluster sampling, systematic sampling, and stratified sampling, although developed for survey work, appear to have application in acceptance inspection. These techniques are presented in books by Deming (29) and Cochran (23).

## Analysis of Single-Stage Inspection Systems Using Explicit Economic Criteria

Methods described in the preceding section in general were developed prior to 1950. They were stimulated by developments in statistical methodology and early application to quality problems by Shewhart (99), Dodge and Romig (32), and others. Abraham Wald's publication (122) in 1950 of a unified theory of decision making based upon statistical evidence gave rise to analyses of inspection operations where economic considerations were explicitly included. The more important of these developments are summarized in this section.

### Choice Between No Inspection and Detail Inspection[8]

Juran (73, pp. 32-33) described economics of sorting operations as follows:

Let $U$ = unit inspection cost,

$K$ = unit cost of failing to detect a defective,

and $p$ = fraction defective of the product.

The decision rule is the following:

(1) If $p < U/K$, do not inspect.

(2) If $p > U/K$, inspect.

This result is illustrated in Figure 3.

The incoming fraction defective, $p$, is interpreted as a parameter of the process which produced the item; that is, $p$ is the probability that an item is produced defective by the process. Sorting and inspection are equally attractive at $p = U/K$, which is called the indifference

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8. Inspection is assumed to be non-destructive.

quality (often denoted by $p_o$). This result is appropriate for continuous inspection or lot-by-lot inspection. Smith (103) arrived at the same expression for indifference quality, while considering lot-by-lot inspection.

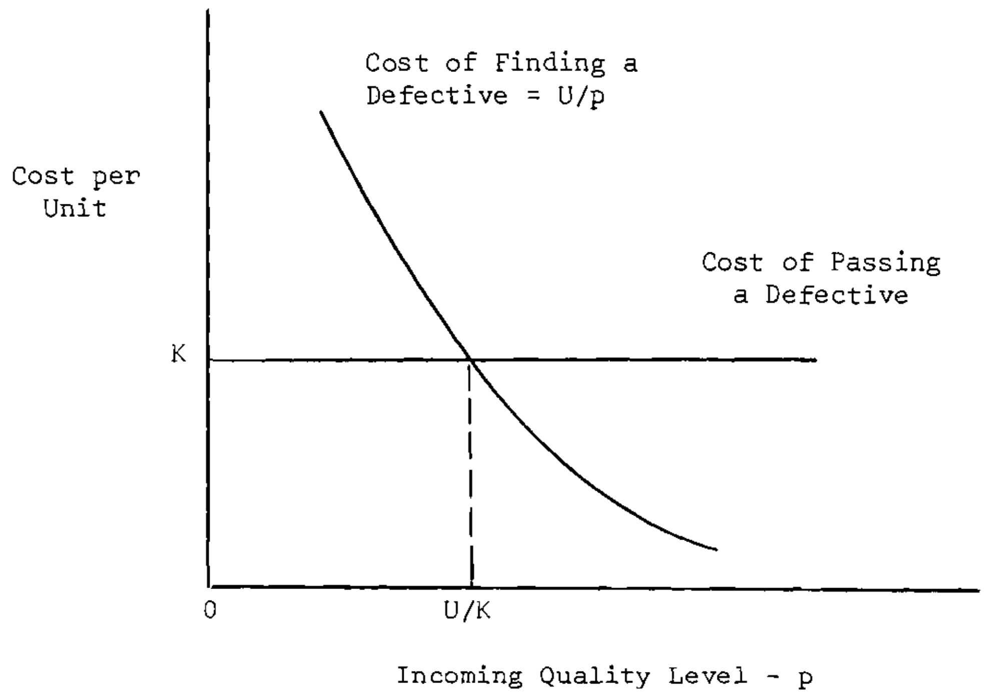


Figure 3. Illustration of Indifference Quality.

## Analysis of Acceptance Sampling

Assuming the validity of the analysis above, it would seem that the decision maker should estimate p and then either sort or use no inspection, depending on the relation of p to $p_o$. However, situations exist where p is not constant and assumes values on both sides of $p_o$. This might be the case for a production process with a changing cause system or in vendor inspection where lots are received from several suppliers. In these situations, sampling inspection is a third alternative to be considered.

General Description. Economic analysis of acceptance sampling requires some assumption about the pattern of variation in the process parameter p. Usually p is assumed to have a probability distribution f(p), called the "process curve." This distribution is utilized in an analysis to select a sampling plan.

An elementary use of cost information and knowledge of the process curve was proposed by Enell (37). He suggests use of a sampling plan whose OC-curve has $p_{.50} = p_o$. Given the sample size code letter,[9] one is to search MIL-STD-105 until he finds a plan having the desired point of control.

A different approach is stated by Anscombe (3):

> The problem is to design an inspection procedure which will minimize
>
> Total cost = Cost of inspection + Decision loss
>
> By "cost of inspection" is meant the whole cost of carrying out the inspection procedure, sampling and testing. By "decision loss" is meant the loss involved in whatever decision is reached by the inspection, as compared with passing goods of perfect quality. . . . To pass goods of perfect quality will in general be the most profitable transaction. To pass goods of less-than-perfect quality, or to reject goods, will in general be a less profitable transaction, the dimunition in profit being called the decision loss.

He goes on to discuss the need for specifying a process curve, an inspection cost curve, and a decision loss curve in order to formulate

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9. Determined by the lot size and choice of inspection level.

a mathematical model for expected cost per lot.

Anscombe's views were shared by many others, especially in Great Britain. Their models differ from one another in detail, but in general they have the following form (illustrated for attribute lot-by-lot inspection):

Let $N$ = lot size,

$p$ = process fraction defective,

$f(p)$ = probability distribution of $p$,

$\delta$ = the decision procedure--the sampling plan,

$S(p,\delta)$ = average cost per lot for sampling and testing,

$L_1(p,\delta,N)$ = loss if a lot of quality $p$ is accepted,

$L_2(p,\delta,N)$ = loss if a lot of quality $p$ is rejected, and

$P_a(p,\delta)$ = probability that a lot of quality $p$ is accepted.

Then the risk associated with inspection of a lot of quality $p$ under a selected decision procedure $\delta$ is given by

$$r(p,\delta) = S(p,\delta) + L_1(p,\delta,N)\, P_a(p,\delta) + L_2(p,\delta,N)\,[1 - P_a(p,\delta)] \cdot \qquad (2\text{-}1)$$

Authors differ in choices of $S$, $L_1$, $L_2$, and approximations used to compute $P_a$. Also they choose varying forms for $\delta$ (single, sequential, etc.). However, Equation (2-1) reflects their general agreement that risk equals inspection cost plus losses attributed to decisions.

Usually the decision function, $\delta$, is to be chosen by either Bayes principle or the minimax principle. Using the Bayes principle of choice, the optimal $\delta = \delta^*$ is chosen to minimize the expected risk, given by

Equation (2-2):

$$R(\delta) = \int_{o}^{1} S(p,\delta)\ f(p)\ dp + \int_{o}^{1} L_1(p,\delta,N)\ P_a(p,\delta)\ f(p)\ dp \qquad (2\text{-}2)$$

$$+ \int_{o}^{1} L_2(p,\delta,N)\ [\ 1 - P_a(p,\delta)]\ f(p)\ dp\ .$$

Then $\delta^*$ satisfies the relation

$$R(\delta^*) = \min R(\delta)\ . \qquad (2\text{-}3)$$

The Bayes procedure requires explicit use of the process curve. Many analysts treat situations where insufficient knowledge exists for complete specification of this function. Their approach is to choose $\delta$ according to the minimax principle; that is, $\delta^*$ satisfies the relation

$$\max_{p} r(\delta,p) = \min_{\delta} \max_{p} r(\delta,p)\ . \qquad (2\text{-}4)$$

Another approach utilized by some analysts is minimization of expected losses from inspection costs and rejection losses, subject to a constraint on the probability of accepting lots of tolerance quality. Horsnell (69) and Davies (28) used this approach because they questioned the ability of a decision maker to evaluate losses from accepting poor quality lots. Specification of the consumer's risk point may be thought of as establishing an "aspiration level" to be met by the plan.

Analysis of Attribute Plans--Bayes Principle. In 1951, Sittig (102), Weibull (123), and Hamaker (59) published papers in which they described approaches utilizing exact knowledge of costs and process curve in order to choose sampling plans.[10] Sittig analyzed several examples utilizing a prior distribution of the Beta family and a loss function linear in the lot fraction defective. He derived optimal plans for both single and double sampling. Weibull suggested the choice of rectifying inspection schemes which have probability 0.50 of accepting lots of indifference quality. In addition, he recommends that the schemes should minimize the average amount of inspection for lots of process average quality, as recommended by Dodge and Romig.

Champernowne (21) used a Beta prior distribution and linear loss functions to derive optimal attribute item-by-item sequential plans.[11] He applied his procedures to three of Sittig's examples and demonstrated (for the same unit inspection cost) that his sequential procedures are more economic than the single sample plans of Sittig.

In 1954, Barnard (6) presented an important paper in which he discussed the role of the prior distribution in determining sampling plans. Using a sequential scheme, linear loss functions, and a two-point mixed binomial prior distribution, he demonstrated that, with different process curves, it is possible to obtain quite different solutions (i.e., plans),

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10. Hamaker referenced two earlier reports by Satterthwaite (91, 92).

11. Readers familiar with sequential procedures may be interested to know that the acceptance and rejection boundaries are not parallel and the acceptance boundary curves upward.

although the assumed cost functions are the same and the process curves have the same mean and variance. In the discussion of this paper, Vagholkar indicated how the two-point mixed binomial distribution could be fitted to more complex distributions--a method yielding an "equivalent" process curve which would be easier to employ. In a later paper (118), he applied his technique to the examples of Sittig and showed that an equivalent mixed binomial with two components yields essentially the same results as the Beta distribution assumed by both Sittig and Champernowne.

Others who have based their analysis of attribute plans on the assumption of complete specification of process curve and loss functions are Suzuki (109), Hamburg (61), Hald (53), Guthrie and Johns (52), Pfanzagl (86), Stevens (107), Vagholkar and Wetherill (119), Schlaifer (95), Raiffa and Schlaifer (88), Wetherill (127), Cox (26), and Smith (103).

Suzuki, Stevens, and Wetherill all analyze single sample plans, with Wetherill assuming a two-component mixed binomial distribution, while the other two writers describe the process curve in general terms. Hamburg, Schlaifer, and Raiffa-Schlaifer use acceptance sampling as a means of illustrating applied statistical decision theory. The Raiffa-Schlaifer book has a large quantity of material on properties of prior distributions and various types of loss functions. The paper by Vagholkar and Wetherill gives a procedure for determining the most economical binomial sequential probability ratio test.

Essentially the same important results regarding asymptotic properties of optimal solutions to single sample plans were obtained independently by Guthrie-Johns and Hald. Using a criterion of minimum expected loss per lot inspected and linear loss functions, they showed that for large lots the sample size is proportional to the square root of the lot size, if the prior distribution is smooth (continuously differentiable) in the neighborhood of the indifference quality, and proportional to the logarithm of the lot size, if the prior distribution assigns probability one to a finite number of points. Guthrie and Johns also indicate that the optimal acceptance number is a linear function of the sample size. The Guthrie-Johns model contains six cost parameters, while Hald uses only three. In a later note Hald demonstrates the equivalence of the two models.[12] The Hald paper contains a comprehensive analysis of the statistical properties of hypergeometric sampling of lots formed from processes having uniform, Polya, Beta, and mixed binomial prior distributions.

Pfanzagl analyzed the results of Hald and found that the optimal sampling procedure is only moderately influenced by small changes in the parameters of the prior distribution. (He assumed a Polya distribution.) He also extended the analysis to double sample schemes and concluded that double sample plans are not often more economic than single sample plans.

Smith investigated problems associated with application of the Guthrie-Johns model. In particular, he was concerned with procedures for obtaining information on costs and the prior distribution, as well as a sensitivity analysis of these factors. He also reported several industrial applications in the electronics industry.

Analysis of Attribute Plans--Minimax Principle. Several authors have preferred the minimax criterion rather than Bayes' principle. In

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12. To be found in *Technometrics*, Vol. 2, 1960, p. 372.

two papers, Breakwell considers minimax procedures for both single sample and sequential schemes. His first paper (17) gives approximate solutions when the acceptable fraction defective is not very small (normal approximation valid); the second paper (18) provides similar results for small fraction defectives (Poisson approximation valid). Ura (116) and Moriguti (85) independently duplicated Breakwell's work. Stevens (107) applied Breakwell's results to obtain a minimax plan for the "windscreen" example of Sittig.

A paper by van der Waerden (120) is unique in that inspection is considered from the point of view of both buyer and seller. The seller may or may not rectify lots while the buyer definitely does not perform rectifying inspection. He applies the minimax criterion to the selection of single sample plans for both cases. An unusual feature is his discussion of a three-person game between buyer, seller and Nature.[13] He shows that, if the producer and buyer form a coalition against Nature (i.e., combine for joint inspection), they will do better economically than if they operate independently.

Anscombe (5) used the minimax criterion to derive a sequential plan for rectifying inspection of lots. He assumed that the number of defectives in the lot follows a Poisson distribution with the distribution of the mean unknown. The loss function is linear. He suggested an acceptance line which is a linear function of the sample size and has an intercept proportional to $K^{-1/2}$ and a slope proportional to $K^{-1}$, where K is the cost of inspecting the entire lot.

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13. Nature is the process producing the lot.

Analysis of Attribute Plans--Constrained Bayes Criterion. Horsnell (69) analyzed nonrectifying inspection, specifying separate loss functions for destructive and nondestructive inspection. Emphasis was on determination of single sample plans, although he considered double sample plans briefly. The criterion was to select the plan which minimized the effective cost per accepted item, subject to the requirement of a fixed consumer's risk point and risk. The effective cost per accepted item was defined as the cost of an accepted lot divided by the expected size of an accepted lot. Horsnell concluded that the optimal sample size is proportional to the logarithm of the lot size when the process curve is binomial. This agrees with Hald's results although the latter minimized expected cost per lot inspected. Optimal sample sizes with non-destructive inspection were found to exceed optimal sample sizes with destructive inspection. Tables are given for many sets of costs and consumer risk points. They are based on a binomial prior distribution.

Several modifications to the Bayes principle were suggested by Johnson (72) in his discussion of methods of choosing a sequential probability ratio test.

Analysis of Variables Plans. Application of economic criteria to the selection of variables plans is not as extensive as in attribute inspection. Breakwell (18, pp. 253-255) found minimax-optimal plans for variables inspection, both with single sampling and with sequential sampling, by using sample data to estimate the lot fraction defective and then appealing to the theories he developed for attribute inspection. He assumed a single normally distributed characteristic with a single upper specification limit.

In his paper on bulk sampling, Duncan (35) determined parameters of a multi-stage sampling procedure by minimizing a cost model, which incorporated sampling costs at the various stages and the gain from using a sampling procedure with a smaller standard error of estimate. He assumed the gain to be a linear function of the reduction in the standard error.

In discussing some statistical considerations of analytical testing in the chemical industry, Davies (28) described the design of a procedure to determine if a batch should be accepted or reprocessed. The cost of rejection was the cost of reprocessing a batch, and the cost of inspection was proportional to the sample size. The product had a single lower specification limit. Measurement errors were normally distributed with mean zero. Knowledge of the prior distribution of lot quality was assumed. No losses were specified for accepted batches which were substandard; rather, the criterion was to choose a sampling plan which minimized expected losses due to sampling and reprocessing, subject to a constraint in the form of a chosen consumer's risk point and risk. Davies solved an example for a normal prior distribution.

Wetherill (128) derived a Bayes solution to the problem of finding a sequential probability ratio test, when a parameter in the distribution of the quality variable has a two-point prior distribution. He also illustrated how this variables procedure could be used to find optimal attribute group sequential sampling plans, provided the group size is large enough for the normal approximation to the binomial distribution to be appropriate.

<u>Analysis of Continuous Sampling Plans</u>. Anscombe (4) has analyzed

the problem of rectifying inspection of a continuous output. He considered the possibilities of no inspection, 100 per cent inspection, and sampling inspection using the Dodge CSP-1 plan. His cost parameters were the unit cost of inspection, the cost of replacing a defective article found in inspection, and the ultimate expected loss due to passing a defective. Utilizing a simple model for the way in which process quality deteriorates, he calculated simple rules for choosing the plan. He suggested that, strictly for the purpose of rectifying inspection (and not for any other purpose such as process control), Dodge's type of sampling inspection plan could hardly be improved on, unless deferred sentencing were permitted.

Economic analysis of continuous inspection, which is concerned with process control as well as with rectification of output, has been pioneered by Girshick and Rubin (41), Savage (93, 94), and Gregory (47).

Evaluation of Economic Analysis of Acceptance Sampling. Efforts to base acceptance sampling decisions on explicit analysis of costs have received criticism because of the difficulty of obtaining information on costs and process curves. For example, Hamaker (60, p. 156) has written the following:

> All in all I am inclined to conclude that the application of such economic theories (minimax included) requires an amount of detailed information that is not as a rule available. They may be successful in isolated cases, but do not lead to simple principles with a wide field of application as primarily needed in industry.

Tippett (111, p. 147) makes the following reply to arguments of this type:

> It seems to me that the attempts that are being made to put acceptance sampling on an economic basis are important. Con-

trol of quality by inspection is important and inspection often adds substantially to manufacturing cost; operating efficiency requires that the economically optimum degree of sampling should be adopted. Moreover, the various decision costs, difficult though some of them are to determine, are closer to the kind of data that industrialists can supply on a rational basis than are the various risks that have been described. It would seem that schemes based on rough estimates of costs, or even informed guesses, are more likely to be satisfactory than those chosen in other ways.

Some other evaluations by people conducting research in this area are the following:

There may be occasions when the less said about economic analysis the better, though it may receive plenty of private thought. Bad (i.e., ordinary) accounting obscures matters, by emphasizing the cost of carrying out the inspection, while hiding the consequences of poor quality of output. . . . it appears that remarkably little economic or other information is of major importance for selecting a good inspection plan, but that small amount is vital.[14]

Recent work by A. Wald on statistical inference starts with the assumption that risks attached to wrong decisions (what I would have called decision losses) can be stated. If that assumption is ever justified, it is, one would suppose, in the field of industrial inspection.[15]

On the positive side decision theory seems to ask the right questions and to solve the right problem. Its use assures a sampling plan that is fully defensible. Most important, statistical decision theory concentrates attention on behavior of the process itself. Only through a thorough understanding of the process curve can the quality control man expect to act economically in improving or controlling quality.[16]

Smith (103, p. 69) summarized an investigation into the applicability of statistical decision theory to the design of acceptance sampling plans with the following statement:

Statistical decision theory has not reached the point where it

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14. Anscombe (4, p. 704).

15. Anscombe (3, p. 67).

16. Smith (103, pp. 68-69).

could or should replace the traditional quality control procedures. Instead it can serve as an important adjunct to these procedures.

## Analysis of Multistage Inspection Systems

Little attention has been given to formal analysis of the interrelationships which exist in multistage production systems. Lieberman (77) analyzed the relationship of the probability of a lot being accepted by all of a system of K inspection stages (certain characteristics inspected at each stage) to the OC-curve of a single-station system (where all characteristics are inspected). Inspection was assumed to be single sampling by attributes.

Schmidt and Sorber (96) considered a problem of determining the most economic point, or points, for screening defective material from a production process. Actually they analyze only a two-station process under fairly limited options. The measure of effectiveness was the cost per hundred pieces considering the cost of inspection at each station and the cost of processing material between the two. Dropouts in production because of defects other than the type being inspected are allowed. The fraction defectives of the production processes are assumed constant.

A dynamic programming approach to determination of optimal levels and locations of screening inspection points in a multistage process was given by Lindsay and Bishop (79). The objective was to minimize the sum of inspection costs and scrap costs, when the requirement for inspection is the maintenance of a specified average outgoing quality. The scrap cost at any stage was defined as manufacturing costs at all stages, plus any costs of disposal. In a brief discussion of an extension of their problem to include costs associated with passed defectives, the authors concluded that screening inspection between any two production stages either should be applied to all items or should be nonexistent. Thus there would be $2^m$ possibilities for a screening inspection system, where m is the number of opportunities for inspection. Under their cost assumptions, dynamic programming methods could be applied to determine the optimal inspection policy.

Heermans (63) analyzed the problem of determining the nature of optimal in-process inspection plans for a multistage production process. He considered attribute single sampling, including degenerate cases of no inspection and detail inspection, and assumed rejected lots were rectified. His cost functions contained terms for inspection and reinspection costs and rework costs, if the defective was not found until final inspection. Application of this technique to a steel tube manufacturing process was claimed. However, close examination of this article reveals that the proposed methods do not adequately account for the interdependcies between successive inspection operations.

Beightler (9, 10) developed a model of a multistation inspection system, which is general enough to allow inclusion of a large number of interactions between stages.[17] The model, for lot-by-lot inspection, is given below:

N = lot size

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17. An apparent limitation is the assumption that a decision at an inspection station will not affect parameters of prior processes.

$W = (w_0, w_1, \ldots, w_N)$, where

$w_j$ = probability of j defectives in a lot reaching the first stage, $j = 0, 1, \ldots, N$

$S_k = (s_{ij}^k)$, a stochastic matrix of dimension $N + 1$, where

$s_{ij}^k$ = probability of transition $(i \to j)$ in number of defectives during inspection at kth station

$R_k = (r_{ij}^k)$, a stochastic matrix of size $N + 1$, where

$r_{ij}^k$ = probability of transition $(i \to j)$ in number of defectives during production at kth operation

$D_k = (d_{ij}^k)$, an $(N + 1) \times (N + 1)$ cost matrix, where

$d_{ij}^k$ = cost associated with a transition $(i \to j)$ in number of defectives during inspection operation k <u>and</u> production operation k.

The expected cost at any stage k is given by

$$C_k = (W \prod_{x=1}^{k-1} S_x R_x) \, [(S_k R_k) \cdot D_k] \,, \tag{2-4}$$

where $(S_k R_k) \cdot D_k$ is a column vector of elements

$$\sum_{j=0}^{N} d_{ij} \, (\sum_{m=0}^{N} s_{im} r_{mj}) \,.$$

The expected cost for the system is given by

$$C = \sum_{k=1}^{n} C_k \,, \tag{2-5}$$

where n is the number of production stages.

The concept of a multistage decision process to design this system was introduced and the problem analyzed by means of the functional equation approach of dynamic programming. The special case, $S_k = S$ and $R_k = R$, for all k, was treated in detail. Two suggested applications were finding optimal $S_1, S_2, \ldots, S_n$ and finding optimal W, given $S_k$ and $R_k$.

## Other Contributions

### Incentive Aspects of Inspection

Hill (64) expresses the view of many analysts who believe that a major benefit of inspection is that it may favorably influence the quality of material offered for acceptance. He believes that the preventive effects of sampling inspection exceed those of detail inspection. The tightened-reduced inspection concept from MIL-STD-414 is evidence of the practical application of this philosophy.

Only one paper was encountered in which an effort was made to incorporate explicitly into a theory the effect of sampling inspection on the quality of lots received. For a constant lot size, Whittle (129) assumes that the process average fraction defective, $\bar{p}$, of lots received depends on the inspection sample size, n, according to

$$\bar{p} = p_u + p_a e^{-cn} , \qquad (2\text{-}6)$$

where $p_u$ is the unavoidable fraction defective inherent in the production process, $p_a$ is the avoidable fraction defective, which can be eliminated by exerting pressure through sampling inspection, and c is a positive constant. Whittle used this relationship to find the optimal allocation of a given inspection effort to a number of different products, for each of which a relation like Equation (2-6) is known to hold.[18]

In discussing the introduction of Whittle's approach into a theory for finding an optimal sample size for a single product, Hamaker (60, p. 150) concludes:

> If we did so, however, our concept of what is "optimum" would have to be drastically revised. For existing theories assume a constant process average or a stationary process curve, and this basic assumption would have to be abandoned. Whittle's principle would justify the maintenance of routine inspection by samples too small to be effective in discriminating good lots from bad. Consequently economic theories so far proposed should be looked upon with some suspicion; it is conceivable that they disregard what is in reality one of the most important functions of inspection procedures: continually to remind people interested that quality does matter.

Inspection Errors

Most formal analysis has been done with the assumption that inspection is accomplished without error. However, there have been some exceptions. Goetz and Johnson (42) incorporated the probability of classifying a defective as good into an economic model for receiving inspection. Different probabilities were used for detail and sampling inspection. Grubbs and Coon (50) describe three criteria for determining test limits relative to specification limits, when screening inspection is subject to measurement error: establish test limits so that the probability of accepting a defective equals the probability of rejecting an effective, or establish test limits to minimize the sum of the two prob-

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18. Whittle used the Lagrange multiplier approach to find the optimal allocation. Kalaba (74) gives the dynamic programming formulation to the same problem.

abilities, or establish test limits to minimize the cost of wrong decisions.

Jackson (70) derives formulas relating the fraction of defective material passed, the fraction of passed material that is defective, the fraction of rejected material that is acceptable, the fraction of material which is actually rejected, and the fraction of material which is actually defective--all which might aid in analysis of the effect of inspection errors in detail inspection.

Cohen (24) recognized the problem of inspectors misclassifying defectives to avoid rejecting a lot. He provides maximum likelihood estimators for the binomial parameter p, when an erroneous report results in recording c defective items when actually there were c + 1. The proportion of such erroneous observations is also estimated, and asymptotic variances and covariances of the estimates are obtained.

<u>Inspection Manpower Requirements</u>

Sespaniak (97) reported an application of queuing theory in determining the optimal number of inspectors for an in-process screening inspection of aircraft engine assemblies. The objective was to minimize the sum of the costs of inspector idle time and inspection-caused assembly line delays.

<u>Least-Cost Testing Sequence</u>

The problem is to determine the least-cost sequence in which to carry out n different nondestructive tests on an item during an inspection operation. If the item is not tested further when it fails a test and if

$C_j$ = cost/item of the jth test,

$R_j$ = probability of rejection on the jth test, and

S = test sequence (some permutation of 1, 2, ..., n),

the objective is to choose S to minimize

$$C(S) = \sum_{j \varepsilon S} C_j \prod_{i<j} (1 - R_i) \quad . \tag{2-7}$$

Price (87), who posed the problem, suggested enumeration of the n! sequences, computation of associated costs, and selection of the minimizing sequence. Mitten (83) and Boothroyd (13) have provided an analytical solution, which is to order the tests in reverse order of the magnitude of the ratio $C_j/R_j$, associated with each test.

## Conclusions

The following conclusions may be drawn from the literature search reported herein:

1. Types of inspection schemes presently available provide adequate choice for the form of an acceptance inspection decision rule.

2. Further development of sampling tables based on noneconomic criteria would be of little value.

3. Efforts to utilize economic criteria and knowledge of the process curve to develop inspection procedures have not resulted in any generally applicable theory.

4. There is no general agreement as to the proper measure of effectiveness for acceptance inspection operations.

5. There is no general agreement as to the appropriate principle of choice in selecting among alternative inspection procedures.

6. Inspection has not been treated adequately as a system of interrelated operations.

7. Successful analysis and design of inspection systems will require a better developed and more clearly understood set of principles--economic, statistical, mathematical, psychological--than now exists relative to acceptance inspection.

# CHAPTER III

# STATISTICAL CONSIDERATIONS IN THE DESIGN OF ACCEPTANCE INSPECTION SYSTEMS

## General

To evaluate alternative decision rules for an acceptance inspection operation, one needs to compute quantities such as the probability of rejecting a lot of any given quality, the expected number of defectives in lots which are accepted, and the average amount of inspection per lot. Computation requires the capability to relate sampling outcomes to the characteristics of the lots and the process by which the lots were generated. The purpose of this chapter is presentation of statistical theories which are necessary if knowledge of prior distributions is to be incorporated into inspection procedures.

Initially, definitions of the various types of probability models will be given--to be followed by treatment of some specific prior distributions. Single sampling is the form of decision rule used to illustrate application of the theory. Simple random sampling is assumed. Primary emphasis is on lot-by-lot inspection by attributes, where the statistic is the number of defectives.[1] Some attention is given to defects-type attribute inspection and to variables inspection.

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1. For a discussion of statistical considerations in continuous inspection, see Savage (93) or Girshick, and Rubin (41).

## Attribute Inspection for Defectives

### Statistical Distributions

A lot of $N$ units is obtained from some process and is submitted for inspection. The lot contains $X$ defective items and $N-X$ good items, where $X$ is a random variable governed by the process of lot generation. Sampling partitions the lot into a sample of $n$ items and the remainder lot of $N-n$ items. Inspection identifies $x$ defectives in the sample; while the number of defectives in the remainder lot, $y = X-x$, remains unknown to the inspector. Under the assumption of simple random sampling without replacement, the conditional distribution of $x$, given $X$, is hypergeometric with parameters $X$, $N$, and $n$. For a given distribution of $X$ (the prior distribution) and hypergeometric sampling, some important properties of $x$ and $y$ may be computed.[2]

Prior Distribution of Lot Quality. The distribution of the number of defectives in the lot will be denoted by $h_N(X)$, where $X = 0, 1, \ldots, N$. The lot fraction defective is $X/N$.

Sampling Distribution. The conditional distribution of $x$, given $X$, is

$$f(x|X) = \begin{cases} \binom{X}{x}\binom{N-X}{n-x} \Big/ \binom{N}{n}, & x = \max(0,n-N+X),\ 1,\ \ldots,\ \min(n,X) \\ 0\ , & \text{otherwise.} \end{cases} \qquad (3\text{-}1)$$

Equation (3-1) may be written in the following equivalent manner:

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2. The following relies heavily on the discussion of the compound hypergeometric distribution given by Hald (53, pp. 291-306).

$$f(x|X) = \frac{\binom{n}{x}\binom{N-n}{X-x}}{\binom{N}{X}} = \frac{\binom{n}{x}\binom{N-n}{y}}{\binom{N}{x+y}} \quad . \tag{3-2}$$

The conditional expectation of x, given X, is

$$E(x|X) = \frac{n\ X}{N} \ . \tag{3-3}$$

The conditional variance of x, given X, is

$$V(x|X) = n\left(\frac{X}{N}\right)\left(\frac{N-X}{N}\right)\left(\frac{N-n}{N-1}\right) \ . \tag{3-4}$$

Joint Distribution of x and X. The joint distribution of x and X is

$$f(x,X) = h_N(X)\ f(x|X) \ . \tag{3-5}$$

This could also be considered as the distribution of x and y, since $X = x + y$. Equation (3-5) could be written

$$f(x,y) = h_N(x+y)\left[\frac{\binom{n}{x}\binom{N-n}{y}}{\binom{N}{x+y}}\right] \ . \tag{3-6}$$

Distribution of the Number of Defectives in the Sample. The marginal distribution of x is obtained by summing f(x,X) over all possible values of X:

$$g_n(x) = \sum_{X=0}^{N} f(x,X) = \binom{n}{x} \sum_{y=0}^{N-n} h_N(x+y) \frac{\binom{N-n}{y}}{\binom{N}{x+y}}, \quad x = 0, 1, \ldots, n. \qquad (3\text{-}7)$$

Hald calls $g_n(x)$ the compound hypergeometric distribution.

Moments of (x,y). The mean and variance of X may be written in the following form:

$$E(X) = N \bar{p} \qquad (3\text{-}8)$$

$$V(X) = N \bar{p} \bar{q} (1 + \delta_N), \quad \delta_N \geq -1 . \qquad (3\text{-}9)$$

Equation (3-8) defines $\bar{p}$, which can be interpreted as the process average fraction defective. In Equation (3-9), $\bar{q} = 1 - \bar{p}$ and $\delta_N$ is a constant which allows comparison of the variance of the prior distribution with that of a binomial distribution having parameters N and $\bar{p}$. The variation of the prior distribution is said to be subnormal if $\delta_N < 0$, normal if $\delta_N = 0$, and hypernormal if $\delta_N > 0$.

The mean and variance of $g_n(x)$ are[3]

3. Derivation of (3-10) through (3-15) may be found in Hald (53, p. 292).

$$E(x) = n\,\bar{p} \tag{3-10}$$

$$V(x) = \frac{n}{N}\left[\frac{n-1}{N-1}\,V(X) + \frac{N-n}{N-1}\,N\bar{p}\bar{q}\right] \quad . \tag{3-11}$$

The mean and variance of the marginal distribution of y are

$$E(y) = (N-n)\,\bar{p} \tag{3-12}$$

$$V(y) = \left[\frac{N-n}{N}\right]\left[\frac{N-n-1}{N-1}\,V(X) + \frac{n}{N-1}\,N\bar{p}\bar{q}\right] \quad . \tag{3-13}$$

The covariance of x and y is

$$COV(x,y) = \frac{n(N-n)}{N(N-1)}\,[V(X) - N\bar{p}\bar{q}] \quad . \tag{3-14}$$

Using Equation (3-9), the covariance may be written as

$$COV(x,y) = n\bar{p}\bar{q}\left(\frac{N-n}{N-1}\right)\delta_N \quad . \tag{3-15}$$

This result was incorporated into an important theorem by Mood (84, p. 417):

> The correlation between the number of defective items in the sample and the number of defectives in the remainder of the lot is positive, zero, or negative according as the variance of X is greater than, equal to, or less than the variance, $N\bar{p}\bar{q}$, of a binomial prior distribution.

Mood's observation is important because it implies that the usual single sample rule, in which rejection occurs for $x > c$, is not appropriate for subnormal prior distributions. For these distributions, a more logical procedure could be to reject for small x and accept for large x.

Hald shows that the variance of x can be written

$$V(x) = n\bar{p}\bar{q}\ (1 + \frac{n-1}{N-1}\,\delta_N) \tag{3-16}$$

which means that $g_n(x)$ is hypernormal (subnormal) if $h_N(X)$ is hypernormal (subnormal). This is also true for the marginal distribution of y.

Conditional Distribution of the Number of Defectives in the Remainder Lot, Given the Number of Defectives in the Sample. The conditional distribution of y for given x is

$$f(y|x) = \frac{f(x,y)}{g_n(x)}\ . \tag{3-17}$$

The mean of this distribution is[4]

$$E(y|x) = (N-n)\left[\frac{(x+1)\ g_{n+1}(x+1)}{(n+1)\ g_n(x)}\right]\ . \tag{3-18}$$

4. Hald (53, p. 293).

$E(y|x)$ is an estimate of the number of defective items in the remainder lot and can be calculated if $g_n(x)$ is given and x is observed.

The variance of y for given x can be calculated from the following expression:

$$V(y|x) = \left[\frac{(N-n)(N-n-1)(x+2)(x+1)\ g_{n+2}(x+2)}{(n+2)(n+1)\ g_n(x)}\right] - E(y|x)\ [E(y|x) - 1]. \quad (3\text{-}19)$$

Derivation of Equation (3-19) is given in Appendix B.

Derivation of Conventional Measures of Effectiveness for Acceptance Sampling Plans for Given Prior Distribution

Three measures of effectiveness commonly have been used to evaluate acceptance sampling plans: the probability of accepting a lot, the expected fraction defective of lots reaching the consumer, and the expected number of units inspected per lot. The first two are related to losses associated with wrong decisions about lots, while the latter is an approximate measure of inspection cost.

Probability of Acceptance. The probability of accepting a lot having a fraction defective X/N is given by

$$P(x \le c|X) = \sum_{x=0}^{c} f(x|X) \quad (3\text{-}20)$$

$$= \sum_{x=0}^{c} \frac{\binom{X}{x}\binom{N-X}{n-x}}{\binom{N}{n}} .$$

If the prior distribution of lot quality is known, the expected proportion of lots which will be accepted is

$$G_n(c) = P(x \le c) = \sum_{x=0}^{c} g_n(x) \quad . \tag{3-21}$$

Bowker and Lieberman (15, p. 404) define the operating characteristic curve of an attribute acceptance sampling plan to be a graph of the probability of accepting a lot as a function of the fraction defective of the lot. It is clear that Equation (3-20) defines their concept of an OC-curve. However, when the prior distribution is known, Equation (3-21) may provide a more useful concept of acceptance probability.

Expected Fraction Defective of Lots Reaching the Consumer. Initially, suppose rejected lots are screened and any defectives found are replaced with good items. If a lot having exactly X defectives is submitted for inspection, the number of defectives ultimately reaching the consumer, D, will be X-x, if $x \le c$, and 0, if $x > c$. Thus, the expected number of defectives remaining in a lot, which initially had X defectives, is

$$E(D|X) = \sum_{x=0}^{c} (X-x) \frac{\binom{X}{x}\binom{N-X}{n-x}}{\binom{N}{n}} \quad . \tag{3-22}$$

If the prior distribution of X is known, E(D|X) can be averaged over all X to yield the expected number of defectives per lot reaching the consumer, E(D):

$$E(D) = \sum_{X=0}^{N} E(D|X)\ h_N(X) . \tag{3-23}$$

An alternate approach is to use Equation (3-18) in the following manner:

$$E(D) = \sum_{x=0}^{c} E(y|x)\ g_n(x) ,$$

which gives

$$E(D) = \frac{N-n}{n+1} \sum_{x=0}^{c} (x+1)\ g_{n+1}(x+1) . \tag{3-24}$$

The average fraction defective of lots reaching the consumer is given by

$$AOQ = \frac{E(D)}{N} . \tag{3-25}$$

If defective items are not replaced when discovered, the average size of a lot reaching the consumer will be less than N. Then the average outgoing quality will be given by

$$AOQ = \frac{E(D)}{N^*} , \tag{3-26}$$

where N* is the expected size of a lot after inspection:

$$N^* = N - \sum_{x=0}^{n} x\, g_n(x) - \sum_{x=c+1}^{n} E(y|x)\, g_n(x) \ .$$

Using Equations (3-10) and (3-18), this can be written as

$$N^* = N - n\bar{p} - \frac{N-n}{n+1} \sum_{x=c+1}^{n} (x+1)\, g_{n+1}(x+1) \ . \tag{3-27}$$

In case rejected lots are not screened, the average number of defectives in a lot reaching the consumer will be

$$E(D) = \frac{\sum_{x=0}^{c} E(y|x)\, g_n(x)}{G_n(c)} = \frac{\left(\frac{N-n}{n+1}\right) \sum_{x=0}^{c} (x+1)\, g_{n+1}(x+1)}{G_n(c)} \ . \tag{3-28}$$

The denominator, $G_n(c)$, is needed to account for the fact that only accepted lots ever reach the consumer.

The average outgoing quality is given by

$$AOQ = \frac{E(D)}{N^*} , \tag{3-29}$$

where

$$N^* = \begin{cases} N\text{-}n, \text{ for destructive testing without replacement} \\ N - \sum_{x=0}^{c} x\, g_n(x) \,/\, G_n(c)\ , \text{ if testing is non-destructive without replacement of defectives} \\ N, \text{ if defectives or destroyed items are replaced with good items} \end{cases} \quad (3\text{-}30)$$

The probability distribution of defectives in lots accepted under the plan is

$$f(y|x \le c) = \sum_{x=0}^{c} f(y|x)\ g_n(x) \,/\, G_n(c)\ . \qquad (3\text{-}31)$$

Average Total Inspection per Lot. For nonrectifying inspection, the average number of units inspected per lot is n. For rectifying inspection, the average total inspection per lot is given by

$$I_n(x) = n + (N\text{-}n)\ [1 - G_n(c)]\ . \qquad (3\text{-}32)$$

Yield per Submitted Lot Under Nonrectifying Inspection. In analyzing an inspection operation where rejected lots are not screened, it is desirable to have a measure of the product yield to the consumer. Here, the chosen measure of yield is the number of items received by the consumer per lot submitted for inspection:

$$Y_n(x) = N^*\ G_n(c)\ , \qquad (3\text{-}33)$$

where $N^*$ is given by Equation (3-30) and the second factor is the expected proportion of lots accepted.

Particular Prior Distributions

Deterministic Prior Distribution. The prior distribution is

$$h_N(X) = \begin{cases} 1 \ , \text{ if } X = A \\ 0 \ , \text{ if } X \neq A \end{cases} \quad . \tag{3-34}$$

This distribution is actually of little value, since it would be an unusual process which generated lots, each having exactly A defective items. (A hypothetical example would be a process where lots are formed of equal portions from the output of ten machines, of which one produces all items defective and the other nine produce all good items.)

For this process

$$g_n(x) = \frac{\binom{A}{x}\binom{N-A}{n-x}}{\binom{N}{n}} \ . \tag{3-35}$$

The number of defectives in the remainder lot is A-x, a deterministic quantity when x is known.

Hypergeometric Prior Distribution. Consider a stock of M items, known to contain A defective items. A lot of size N (< M) items is selected at random from this stock. The distribution of X is

$$h_N(X;A,M) = \begin{cases} \binom{A}{X}\binom{M-A}{N-X}\Big/\binom{M}{N}, & X = 0, 1, \ldots, A \\ 0, \text{ elsewhere} \end{cases} \tag{3-36}$$

with mean and variance

$$E(X) = N\,\bar{p} \tag{3-37}$$

$$V(X) = N\bar{p}\bar{q}\left(\frac{M-N}{M-1}\right), \tag{3-38}$$

where $\bar{p} = A/M$ is the process average fraction defective. This distribution is subnormal because

$$\delta_N = -\frac{N-1}{M-1} < 0 .$$

This means that x and y are negatively correlated.

The joint probability distribution of x and y, defined by Equation (3-6), can be written as

$$f(x,y) = h_n(x;A,M)\; h_{N-n}(y;A-x,M-n) . \tag{3-39}$$

The marginal distribution of x is therefore

$$g_n(x) = h_n(x;A,M) = \begin{cases} \dfrac{\binom{A}{X}\binom{M-A}{n-x}}{\binom{M}{n}}, & x = 0,\ 1,\ \ldots,\ n \\ 0, & \text{elsewhere} \end{cases} \tag{3-40}$$

and the conditional distribution of y, given x, is

$$\begin{aligned} f(y|x) &= h_{N-n}(y;A-x,M-n) \\ &= \frac{\binom{A-x}{y}\binom{M-n-A+x}{N-n-y}}{\binom{M-n}{N-n}}, \text{ for } y = 0,\ 1,\ \ldots,\ A-x\ . \end{aligned} \tag{3-41}$$

The mean of $f(y|x)$ is

$$E(y|x) = (n-n)\ \frac{A-x}{M-n}\ , \tag{3-42}$$

which is a decreasing function of x.

Hald's Theorem on Distributions Reproducible Under Hypergeometric Sampling. The results stated above for a hypergeometric prior distribution and much of what follows in this section are based upon a theorem by Hald (53, p. 299):

> Let X denote the number of elements having a certain attribute in a population of N elements and let x and y = X-x denote the corresponding numbers of elements in a random sample (drawn without replacement) of size n and in the remainder of the population, respectively. If the distribution of X is a hypergeometric, a binomial, a rectangular, a Polya, or a mixed binomial distribution, or any weighted average of these distributions with weights independent of N and X, then for any N the

distribution of x is the same as the distribution of X with n substituted for N, and the distribution of y for given x is also of the same type but with parameters depending on x and n.

Thus, for the named distributions, $g_n(x) = h_n(x)$. Fortunately, most of the prior distributions appearing in the literature of acceptance sampling are included among those listed by Hald.

Binomial Prior Distribution. Consider a process which produces items with a constant probability, $\bar{p}$, of being defective. Lots of size N are formed from the output. The distribution of X is

$$h_N(X;\bar{p}) = \binom{N}{X} \bar{p}^X \bar{q}^{N-X}, \quad X = 0, 1, \ldots, N \,, \tag{3-43}$$

with mean and variance

$$E(X) = N \bar{p} \tag{3-44}$$

$$V(X) = N \bar{p} \bar{q} \ . \tag{3-45}$$

The distribution has normal dispersion, since $\delta_N = 0$. This means that x and y are uncorrelated (moreover, they are independent random variables).

The joint probability distribution of x and y can be written as:

$$f(x,y) = h_n(x;\bar{p}) \, h_{N-n}(y;\bar{p}) \ . \tag{3-46}$$

Therefore,

$$g_n(x) = h_n(x;\bar{p}) = \binom{n}{x} \bar{p}^x \bar{q}^{n-x} \quad , \; x = 0, 1, \ldots, n \qquad (3\text{-}47)$$

and

$$f(y|x) = f(y) = h_{N-n}(y;\bar{p}) \qquad (3\text{-}48)$$

$$= \binom{N-n}{y} \bar{p}^y \bar{q}^{N-n-y} \quad , \; y = 0, 1, \ldots, N\text{-}n.$$

The expected number of defectives in the remainder lot is independent of x:

$$E(y|x) = E(y) = (N\text{-}n)\ \bar{p} \ . \qquad (3\text{-}49)$$

Polya Prior Distribution.[5] Consider a production process which generates a lot in the following manner: the probability of producing the first item defective is $\bar{p}$ and this probability changes with each item produced such that after having produced (a+d) items of which d are defective the probability that the next item is defective is given by

$$\frac{\bar{p} + d\ r}{1 + (a+d)\ r} , \quad r > -1 \quad . \qquad (3\text{-}50)$$

The probability of a defective is assumed to be a linear function of the

5. Discussed by Feller (40, p. 110), Hald (53, p. 297), and Pfanzagl (86).

number of defectives previously generated by the process. The distribution of X is

$$h_N(X;\bar{p},r) =$$

$$\binom{N}{X} \frac{\bar{p}(\bar{p}+r)\cdots(\bar{p}+(X-1)r)\bar{q}(\bar{q}+r)\cdots(\bar{q}+(N-X-1)r)}{1(1+r)\cdots(1+(N-1)r)}\,, \tag{3-51}$$

$$\text{for } X = 0, 1, \ldots, N\,.$$

This may also be written as:

$$h_N(X;s,t) = \binom{N}{X} \frac{\Gamma(s+X)\Gamma(t+N-X)\Gamma(s+t)}{\Gamma(s)\ \Gamma(t)\ \Gamma(s+t+N)}\,, \tag{3-52}$$

where the relation between Equations (3-51) and (3-52) is $s = \bar{p}/r$ and $t = \bar{q}/r$, or $r = 1/(s+t)$ and $\bar{p} = s/(s+t)$, $\bar{q} = t/(s+t)$.

The Polya distribution has mean and variance

$$E(X) = N\bar{p} = N\,\frac{s}{s+t} \tag{3-53}$$

$$V(X) = N\bar{p}\bar{q}\left(\frac{1+Nr}{1+r}\right)$$

$$= N\,\frac{st}{(s+t)^2}\left(\frac{s+t+N}{s+t+1}\right)\,. \tag{3-54}$$

For this distribution

$$\delta_N = (N-1) \quad \frac{r}{1+r} \tag{3-55}$$

indicating that the Polya distribution is hypernormal for $r > 0$ and subnormal for $r < 0$.

The joint distribution of x and X may be written[6]

$$\left.\begin{aligned} f(x,y) &= h_n(x;\bar{p},r)\ h_{N-n}(y;\bar{p}_x,r_n) \\ \text{where} \qquad \bar{p}_x &= \frac{\bar{p} + rx}{1 + nr} \\ r_n &= \frac{r}{1 + nr} \quad . \end{aligned}\right\} \tag{3-56}$$

From Equation (3-56) it is seen that

$$g_n(x) = h_n(x;\bar{p},r) \tag{3-57}$$

and

$$f(y|x) = h_{N-n}(y;\bar{p}_x,r_n) \quad . \tag{3-58}$$

The mean of $f(y|x)$ is

6. Hald (53, p. 298).

$$E(y|x) = \bar{p}_x = \frac{\bar{p} + xr}{1 + nr} = \frac{s + x}{s + t + n} , \tag{3-59}$$

which is an increasing function of x if $h_N(X)$ is hypernormal and a decreasing function of x in the subnormal case.

Special Cases of the Polya Distribution. Three special cases of the Polya distribution may be recognized:

1. Binomial distribution: $r = 0$.
2. Hypergeometric distribution: $r = -1/M$.
3. Uniform discrete distribution: $r = \bar{p} = \bar{q} = 1/2$.

The uniform distribution is given by

$$h_N(X) = \frac{1}{N+1} , \text{ for } X = 0, 1, \ldots, N . \tag{3-60}$$

Mixed Binomial Distribution. Consider a process wherein the probability of a defective remains constant during the generation of a lot, but varies from lot to lot in accordance with a given weight function. The distribution of X would be (for m possible process levels):

$$\left. \begin{aligned} & h_N(X;p_i,w_i) = \sum_{i=1}^{m} w_i \binom{N}{X} p_i^{\,X} q_i^{N-X}, \quad X = 0, 1, \ldots, N \\ & \text{where} \quad \sum_{i=1}^{m} w_i = 1 , \text{ and } w_i \geq 0, \; i = 1, 2, \ldots, m. \end{aligned} \right\} \tag{3-61}$$

The mean is

$$\left.\begin{aligned} E(X) &= n\,\bar{p} \\ \text{where} \qquad \bar{p} &= \sum_{i=1}^{m} w_i p_i \quad . \end{aligned}\right\} \tag{3-62}$$

The variance is

$$V(X) = N \sum_{i=1}^{m} w_i p_i q_i + N^2 \sum_{i=1}^{m} w_i (p_i - \bar{p})^2 \quad . \tag{3-63}$$

The mixed binomial distribution is hypernormal, since

$$\delta_N = (N-1) \sum_{i=1}^{m} \frac{w_i (p_i - \bar{p})^2}{\bar{p}\bar{q}} \tag{3-64}$$

which is non-negative.

The joint distribution of x and X can be factored to give[7]

$$f(x,X) = h_n(x;p_i,w_i)\; h_{N-n}(y;p_i,w_i(x)) \tag{3-65}$$

where

$$w_i(x) = \frac{w_i p_i^x q_i^{n-x}}{\sum_{i=1}^{m} w_i p_i^x q_i^{n-x}} \quad . \tag{3-66}$$

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7. Hald (53, p. 298).

From Equation (3-65) it may be seen that

$$g_n(x) = h_n(x;p_i,w_i) = \sum_{i=1}^{m} w_i \binom{N}{X} p_i^x q_i^{n-x} \qquad (3\text{-}67)$$

and

$$f(y|x) = h_{N-n}(y;p_i,w_i(x))$$

$$= \sum_{i=1}^{m} w_i(x) \binom{N-n}{y} p_i^y q_i^{N-n-y} \ . \qquad (3\text{-}68)$$

The expected number of defectives in the remainder lot can be computed from

$$E(y|x) = (N-n) \sum_{i=1}^{m} w_i(x)\, p_i \ . \qquad (3\text{-}69)$$

Mixed Binomial Distribution with Continuous Weight Function. The process fraction defective might be considered a random variable with probability distribution w(p), such that

$$\int_0^1 w(p) = 1 \ .$$

Incorporating this into the mixed binomial distribution, one obtains

$$h_N(X;w(p)) = \binom{N}{X} \int_0^1 p^X q^{N-X} w(p)\, dp \;. \tag{3-70}$$

The process average fraction defective, $\bar{p}$, would be

$$E(p) = \int_0^1 p\, w(p)\, dp \tag{3-71}$$

Several analysts[8] have chosen a Beta distribution for the weight function, $w(p)$:

$$w(p) = \frac{\Gamma(s+t)}{\Gamma(s)\,\Gamma(t)}\, p^{s-1}\,(1-p)^{t-1} \;,\; s > 0,\; t > 0 \;. \tag{3-72}$$

Using Equation (3-70),

$$h_N(X) = \binom{N}{X} \frac{\Gamma(s+X)\,\Gamma(t+N-X)\,\Gamma(s+t)}{\Gamma(s)\,\Gamma(t)\,\Gamma(s+t+N)} \;. \tag{3-73}$$

Comparison with Equation (3-52) reveals this to be a Polya Distribution.

## Attribute Inspection for Defects

### Statistics and Probability Distributions

A single sampling plan for defects requires selection of a random sample of n units from the lot of N units, determination of the total

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8. Smith (103, p. 21) and Hald (53, p. 299).

number of defects, x, in the sample, and acceptance of the lot if x does not exceed some chosen non-negative integer, c.[9] The following statistics and their associated probability distributions are relevant to the analysis of this procedure.

Distribution of the Number of Defects in a Unit. Assume that units are produced by a process having an intensity of $\lambda$. The parameter $\lambda$ is the average number of defects per unit produced by the process. Let d be the number of defects in a given unit. For fixed $\lambda$, d is a random variable having probability distribution $f(d|\lambda)$, defined for $d = 0, 1, \ldots$, and $\lambda > 0$. By definition, $E(d|\lambda) = \lambda$. The Poisson distribution is universally used to represent $f(d|\lambda)$. It is given by

$$f(d|\lambda) = \frac{e^{-\lambda}\lambda^{d}}{d!}, \; d = 0, 1, \ldots \tag{3-74}$$

Distribution of the Process Parameter $\lambda$. The prior distribution of the process parameter $\lambda$ will be denoted by $h(\lambda)$, which could be continuous or discrete. The expected value of $\lambda$ will be denoted by $\bar{\lambda}$.

Distribution of the Number of Defects in a Sample, for Given $\lambda$. For a random sample of n units, let x be the total number of defects found. Thus,

$$x = d_1 + d_2 + \cdots + d_n \tag{3-75}$$

9. A unit of product could be arbitrary, as in the case of the unit being a 100-yard piece of cloth.

and, if $\lambda$ is assumed constant during the formation of the lot,

$$f_n(x|\lambda) = \frac{e^{-n\lambda}(n\lambda)^x}{x!}, \quad x = 0, 1, 2, \ldots \tag{3-76}$$

The last result follows from the fact that the distribution of the sum of n independent Poisson variables, each with mean $\lambda$, is also Poisson, but with mean $n\lambda$.[10] The variance of x is also $n\lambda$.

Joint Distribution of x and $\lambda$. The joint distribution of the number of defects in the sample and the process intensity $\lambda$ is given by

$$f(x,\lambda) = f_n(x|\lambda)\, h(\lambda) = h(\lambda) \frac{e^{-n\lambda}(n\lambda)^x}{x!}. \tag{3-77}$$

Marginal Distribution of x. The distribution $f_n(x|\lambda)$ averaged over all $\lambda$ yields the marginal distribution of x:

$$g_n(x) = \begin{cases} \sum_{\lambda} h(\lambda) \frac{e^{-n\lambda}(n\lambda)^x}{x!}, & \text{if } \lambda \text{ is discrete} \\ \int_0^{\infty} h(\lambda) \frac{e^{-n\lambda}(n\lambda)^x}{x!}\, d\lambda, & \text{if } \lambda \text{ is continuous} \end{cases} \quad )3\text{-}78)$$

which has mean and variance:[11]

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10. Raiffa and Schlaifer (88, p. 283).

11. Appendix C.

$$E(x) = n\bar{\lambda} \tag{3-79}$$

$$V(x) = n^2 V(\lambda) + n\bar{\lambda} \ . \tag{3-80}$$

Conditional Distribution of $\lambda$ for a Given $x$. The conditional distribution of $\lambda$ when $x$ has been observed is

$$f(\lambda|x) = \frac{f(x,\lambda)}{g_n(x)} \ , \ g_n(x) \neq 0 \ . \tag{3-81}$$

The mean of this distribution is

$$E(\lambda|x) = \begin{cases} \sum\limits_{\lambda} \lambda f(\lambda|x) \\ \\ \displaystyle\int_0^{\infty} \lambda f(\lambda|x) d\lambda \end{cases} \tag{3-82}$$

## Results for Particular Prior Distributions

Results have been obtained for two particular prior distributions: a gamma distribution and an m-point discrete distribution.

Gamma Prior Distribution.[12] The process intensity $\lambda$ has probability density function

$$h(\lambda;a,b) = \frac{b^a}{(a-1)!} \ e^{-b\lambda}\lambda^{a-1} \ , \ \lambda > 0; \ a, \ b > 0. \tag{3-83}$$

For this distribution:

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12. The results of this paragraph are supported by derivations in Raiffa and Schlaifer (88, pp. 283, 284).

$$E(\lambda) \equiv \bar{\lambda} = \frac{a}{b} \tag{3-84}$$

$$V(\lambda) = \frac{a}{b^2} . \tag{3-85}$$

The joint distribution of $\lambda$ and x is

$$f(x,\lambda) = \frac{b^a}{(a-1)!} e^{-b\lambda}\lambda^{a-1}e^{-n\lambda} \frac{(n\lambda)^x}{x!} . \tag{3-86}$$

The marginal distribution of x is

$$g_n(x) = \int_0^\infty f(x,\lambda)\, d\lambda$$

$$= \frac{(a+x-1)!}{x!\ (a-1)!} \left(\frac{n}{n+b}\right)^x \left(\frac{b}{n+b}\right)^a . \tag{3-87}$$

The mean and variance of x are

$$E(x) = \frac{na}{b} = n\bar{\lambda} \tag{3-88}$$

$$V(x) = \frac{(n+b)na}{b^2} . \tag{3-89}$$

Note that $g_n(x)$ is a negative binomial distribution with parameters a and n/(n+b).

The conditional distribution of $\lambda$, given x, is gamma with parameters (a+x) and (b+n):

$$f(\lambda|x) = h(\lambda;a+x,b+n)$$

$$= \frac{(b+n)^{a+x}}{(a+x-1)!} e^{-(b+n)\lambda} \lambda^{(a+x-1)} \quad , \lambda > 0. \tag{3-90}$$

The mean and variance are

$$E(\lambda|x) = \frac{a + x}{b + n} \tag{3-91}$$

$$V(\lambda|x) = \frac{(a+x)}{(b+n)^2} . \tag{3-92}$$

m-Point Discrete Distribution. The process intensity has probability distribution

$$h(\lambda;w_i) = \begin{cases} w_i, \text{ if } \lambda = \lambda_i, \ i = 1, 2, \ldots, m \\ 0 \ , \text{ otherwise.} \end{cases} \tag{3-93}$$

For this distribution

$$E(\lambda) \equiv \bar{\lambda} = \sum_{i=1}^{m} w_i\lambda_i \tag{3-94}$$

$$V(\lambda) = \sum_{i=1}^{m} w_i \lambda_i^2 - \bar{\lambda}^2 . \tag{3-95}$$

The joint distribution of x and $\lambda$ is

$$f(x,\lambda) = w_i \frac{e^{-n\lambda_i}(n\lambda_i)^x}{x!} , \ i = 1, 2, \ldots, m . \tag{3-96}$$

The marginal distribution of x is

$$g_n(x) = \sum_{i=1}^{m} w_i \frac{e^{-\lambda_i n}(n\lambda_i)^x}{x!} , \tag{3-97}$$

which has mean $n\bar{\lambda}$ and variance[13]

$$V(x) = \sum_{i=1}^{m} n^2 w_i \lambda_i^2 - n\bar{\lambda}(n\bar{\lambda}-1) . \tag{3-98}$$

The conditional distribution of $\lambda$, given x, is an m-point discrete distribution[14]

$$f(\lambda|x) = h(\lambda;w_i(x))$$

$$= \begin{cases} w_i(x) , & \text{if } \lambda=\lambda_i, i=1,2,\ldots,m \\ 0, & \text{otherwise} \end{cases} \tag{3-99}$$

13. See Appendix C.2.

14. See Appendix C.3.

where

$$w_i(x) = \frac{w_i e^{-n\lambda_i}(n\lambda_i)^x}{\sum_{i=1}^{m} w_i e^{-n\lambda_i}(n\lambda_i)^x} . \quad (3\text{-}100)$$

For this distribution[15]

$$E(\lambda|x) = \sum_{i=1}^{m} w_i(x)\ \lambda_i \quad (3\text{-}101)$$

$$= \frac{(x+1)}{n}\ \frac{g_n(x+1)}{g_n(x)}$$

$$V(\lambda|x) = \frac{(x+1)}{[ng_n(x)]^2}\,[(x+2)g_n(x+2)g_n(x) - (x+1)g_n(x+1)] . \quad (3\text{-}102)$$

Characteristics of a Single Sample Plan (n,c)

Probability of Accepting a Lot. When the prior distribution is given, the probability of a lot passing the inspection plan is

$$G_n(c) = P(x \le c) = \sum_{x=0}^{c} g_n(x) . \quad (3\text{-}103)$$

Average Total Inspection per Lot for Rectifying Inspection. The expected number of units inspected per lot submitted under rectifying inspection is

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15. See Appendix C.4.

$$I_n(x) = n + (N-n)\,[1 - G_n(c)]\ . \tag{3-104}$$

Expected Number of Defects in the Remainder Lot. The N-n units which are not inspected may contain defects. Given $g_n(x)$ and having observed x, one can compute the posterior estimate of $\lambda$ from Equation (3-82). Denoting this estimate by $\lambda_x$ and letting y be the number of defects in the remainder lot:

$$E(y|x) = \sum_{y=0}^{\infty} y\,\frac{e^{-(N-n)\lambda_x}[(N-n)\lambda_x]^y}{y!} = (N-n)\lambda_x\ . \tag{3-105}$$

An alternate method for computing $E(y|x)$ is to utilize the distribution of the number of defects in the remainder lot, conditioned on the observed x:

$$f(y|x) = \begin{cases} \sum\limits_{\lambda} f(\lambda|x)e^{-(N-n)\lambda}[(N-n)\lambda]^y/y!\ , & \text{if } \lambda \text{ is discrete} \\ \\ \int_0^{\infty} f(\lambda|x)e^{-(N-n)\lambda}[(N-n)\lambda]^y/y!\ , & \text{if } \lambda \text{ is continuous.} \end{cases} \tag{3-106}$$

From this distribution,[16]

$$E(y|x) = \sum_{y=0}^{\infty} yf(y|x) = (N-n)\lambda_x\ . \tag{3-107}$$

16. Appendix C.5.

Expected Number of Defects in Accepted Lots. The probability distribution of the number of defects in accepted lots is given by

$$f(y|x \le c) = \sum_{x=0}^{c} \frac{f(y|x)\, g_n(x)}{G_n(c)}, \qquad (3\text{-}108)$$

which has mean

$$E(y|x \le c) = \sum_{y=0}^{\infty} y f(y|x \le c)$$

$$= (N-n) \sum_{x=0}^{c} \lambda_x \frac{g_n(x)}{G_n(c)} . \qquad (3\text{-}109)$$

Defects found in the sample are assumed to be repaired before the lot is sent to the consumer.

Expected Number of Defects in Lots Reaching the Consumer When Rectifying Inspection is Used. If rejected lots are screened and all defects repaired, including those found in the sample, the only source of defects reaching the consumer will be in the uninspected portions of accepted lots. Therefore the average number of defects per lot reaching the consumer will be

$$E(y|x \le c)\, G_n(c) = (N-n) \sum_{x=0}^{c} \lambda_x g_n(x) . \qquad (3\text{-}110)$$

Expected Yield when Nonrectifying Inspection is Used. When only accepted lots reach the consumer, the number of units received per lot submitted for inspection is

$$Y_n(x) = N\ G_n(c)\ . \qquad (3\text{-}111)$$

In deriving Equation (3-111) it was assumed that no units are removed from the samples from accepted lots.

## Variables Inspection

### Statistics and Probability Distributions

If the desirability of accepting a lot is a function of the mean value of a single measurable quality characteristic and if the variability of this characteristic is known, a single sample plan by variables would require selection of a random sample of n items, measurement of $x_1, x_2, \ldots, x_n$, and acceptance of the lot if $a_1 \le m \le a_2$, where m is the sample mean and $a_1$ and $a_2$ are parameters of the plan. The following statistics and their associated probability distributions are relevant to the analysis of this procedure.

Distribution of an Individual Measurement. Assume that items are produced by a process having a mean of $\mu$. The parameter $\mu$ is the expected value of the measurement, x, on an individual item. For fixed $\mu$, x is a random variable having probability distribution $f(x|\mu)$. Assume that this distribution is normal with variance $\sigma^2$. Thus,

$$f(x|\mu,\sigma) = (2\pi\sigma^2)^{-1/2} \exp\left[\frac{-(x-\mu)^2}{2\sigma^2}\right] . \qquad (3\text{-}112)$$

Distribution of the Process Parameter $\mu$. The prior distribution of the process parameter $\mu$ will be denoted by $h(\mu)$. The expected value of $\mu$ is $\bar{\mu}$.

Distribution of the Sample Mean for a Given $\mu$. For a random sample of n items, let m be the mean of the observed measurements:

$$m = \frac{1}{n} (x_1 + x_2 + \ldots + x_n) . \qquad (3\text{-}113)$$

If $\mu$ is assumed constant during formation of the lot,

$$f_n(m|\mu,\sigma) = \left[\frac{n}{2\pi\sigma^2}\right]^{\frac{1}{2}} \exp\left[\frac{-n(m-\mu)^2}{2\sigma^2}\right] . \qquad (3\text{-}114)$$

The sample mean is a normally distributed random variable with expected value $\mu$ and variance $\sigma^2/n$.

Joint Distribution of m and $\mu$. The joint distribution of the sample mean and the process mean is

$$f(m,\mu) = f_n(m|\mu,\sigma)\ h(\mu) . \qquad (3\text{-}115)$$

Marginal Distribution of m. The function $f_n(m|\mu,\sigma)$ averaged over all $\mu$ yields the marginal distribution of m:

$$g_n(m;\sigma) = \int_{-\infty}^{\infty} f_n(m|\mu,\sigma)\ h(\mu)\ d\mu \qquad (3\text{-}116)$$

This distribution has moments[17]

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17. Appendix D.

$$E(m) = \bar{\mu} \tag{3-117}$$

$$V(m) = V(\mu) + \frac{\sigma^2}{n} \ . \tag{3-118}$$

Conditional Distribution of $\mu$ for a Given m. The conditional distribution of $\mu$ when m has been observed is

$$f(\mu|m) = \frac{f(m,\mu)}{g_n(m;\sigma)} \ . \tag{3-119}$$

This distribution has expected value

$$E(\mu|m) = \int_{-\infty}^{\infty} \mu f(\mu|m) \, d\mu \ . \tag{3-120}$$

Results for a Normal Prior Distribution

In this section, $\mu$ is assumed to be normally distributed with parameters $\bar{\mu}$ and v:

$$h(\mu) = (2\pi v)^{-\frac{1}{2}} \exp\left[\frac{-(\mu-\bar{\mu})^2}{2v}\right] \ . \tag{3-121}$$

The parameter v is the variance of $\mu$.

The joint distribution of m and $\mu$ is

$$f(m,\mu) = (2\pi\sigma)^{-1} \left(\frac{n}{v}\right)^{\frac{1}{2}} \exp\left[\frac{-(\mu-\bar{\mu})^2}{2v} - \frac{n(m-\mu)^2}{2\sigma^2}\right]. \tag{3-122}$$

The marginal density of m is[18]

$$g_n(m;\sigma) = \int_{-\infty}^{\infty} f(m,\mu)\, d\mu \qquad (3\text{-}123)$$

$$= (2\pi)^{-\frac{1}{2}}(v+\sigma^2/n)^{-\frac{1}{2}} \exp\left[\frac{-(m-\bar{\mu})^2}{2(v+\sigma^2/n)}\right].$$

Thus, $g_n(m)$ is normal with mean and variance

$$E(m) = \bar{\mu} \qquad (3\text{-}124)$$

$$V(m) = v + \sigma^2/n \; . \qquad (3\text{-}125)$$

The conditional distribution of $\mu$, given m, is[19]

$$f(\mu|m) = (2\pi)^{-\frac{1}{2}}\left(\frac{v\sigma^2}{\sigma^2+nv}\right)^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\left(\mu-\frac{\bar{\mu}\sigma^2+nv}{\sigma^2+n\mu}\right)^2\left(\frac{v\sigma^2}{\sigma^2+nv}\right)^{-1}\right], \qquad (3\text{-}126)$$

which has mean and variance

$$E(\mu|m) = \frac{\bar{\mu}\sigma^2+nv}{\sigma^2+nv} \qquad (3\text{-}127)$$

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18. Raiffa and Schlaifer (88, p. 297).

19. Raiffa and Schlaifer (88, p. 295).

$$V(\mu|m) = \frac{v\sigma^2}{\sigma^2+nv} \quad . \tag{3-128}$$

Characteristics of the Single Sample Plan $(n,a_1,a_2)$

Probability of Accepting a Lot. When the prior distribution is given, the probability of a lot passing the inspection plan is

$$P_a = G_n(a_2) - G_n(a_1) = \int_{a_1}^{a_2} g_n(m;\sigma)\ dm \tag{3-129}$$

where $G_n(m)$ is the cumulative distribution function of m.

Average Total Inspection per Lot for Rectifying Inspection. When the lot consists of N items and rejected lots are rectified, the average number of units inspected per lot is given by

$$I_n(m) = n + (N-n)(1-P_a) \ . \tag{3-130}$$

Expected Yield When Nonrectifying Inspection is Used. When only accepted lots reach the consumer, the expected number of units received per lot submitted for inspection is

$$Y_n(m) = N^* P_a \quad , \tag{3-131}$$

where N* is the average size of the lot after inspection and is given by

$$N^* = \begin{cases} N-n, & \text{if testing is destructive} \\ N, & \text{if testing is not destructive.} \end{cases} \tag{3-132}$$

Expected Fraction Defective in the Remainder Lot. When the specifications on the quality characteristic x are (L,U) and m has been observed, an estimate of the fraction defective in the remainder lot is given by

$$p_{N-n}(m) = 1 - \int_{L}^{U} (2\pi)^{-\frac{1}{2}} \sigma^{-1} \exp\left[\frac{-(x-\mu_m)^2}{2\sigma^2}\right] dx \;, \qquad (3\text{-}133)$$

where $\mu_m$ is the mean of the posterior distribution $f(\mu|m)$ and is defined by Equation (3-120).

Expected Fraction Defective in Accepted Lots. If defectives found in the sample are repaired or replaced with effectives, the estimated fraction defective in accepted lots is

$$\left(\frac{N-n}{N}\right) \int_{a_1}^{a_2} p_{N-n}(m)[g_n(m) \;/\; P_a] \; dm \;. \qquad (3\text{-}134)$$

Estimated Fraction Defective Reaching the Consumer when Rectifying Inspection is Used. If rejected lots are screened and if any defectives found in inspection are replaced and repaired, the only source of defectives reaching the consumer will be the uninspected portion of accepted lots. Therefore, the average fraction defective of lots after inspection is given by

$$AOQ = \left(\frac{N-n}{N}\right) \int_{a_1}^{a_2} p_{N-n}(m) \; g_n(m) \; dm \;. \qquad (3\text{-}135)$$

## Summary

In this chapter, procedures have been developed which permit knowledge of the distribution of lot or process parameters to be incorporated into calculation of commonly used measures of effectiveness for acceptance sampling plans. The results were illustrated for single sample plans for the following types of inspection:

1. Attribute inspection for defectives.
2. Attribute inspection for defects.
3. Variables inspection with known process standard deviation.

Several prior distributions were considered for attribute inspection and all had the important property of reproducibility under hypergeometric sampling--that is, the marginal distribution of the sample statistic was of the same form as the prior distribution of the lot parameter. The results for the mixed binomial distribution are valuable because any prior distribution can be approximated with it.

While the results for attribute inspection for defectives followed the work of Hald (54), the developments reported for defects inspection and variables inspection are believed to represent a new adaptation of statistical principles to acceptance inspection. Some particular prior distributions were analyzed for these latter two types of inspection.

It is shown in the next chapter that the ability to utilize knowledge of the prior distribution and information from sampling to make predictions about the remainder lot is of great value in decision making. In this context, it should be pointed out that it is not at all clear how standard inspection procedures (Dodge-Romig, Military Standards) utilize this information. It is hoped that future inspection standards will be developed with a clearer description of process characteristics in mind.

# CHAPTER IV

# ECONOMIC BASES FOR THE DESIGN OF ACCEPTANCE INSPECTION SYSTEMS

## General

The purpose of this chapter is to present fundamental principles of decision making based upon economic criteria. Initially, the basic elements of decision theory are described and extended to include statistical decisions. This is followed by a discussion of the various gains and losses to the firm which are influenced by acceptance inspection decisions. Attention is given to quantification of these gains and losses in monetary terms. Next, appropriate measures of effectiveness for inspection decisions are determined and several principles of choice for selecting among alternatives are evaluated. Finally, the economic implications of the more important sampling schemes are analyzed.

## Decision Theory

### Formulation of a Decision Problem with No Data

Components of a Decision Problem. The decision maker's problem is that he has certain objectives, which he desires to attain or retain, various alternative ways of attempting to achieve these objectives, and, in general, uncertain knowledge about future events which influence the success with which an alternative satisfies the objectives. The analysis of decision problems is facilitated if the problem is made explicit through the following formal structure:

1. The set of objectives of the decision maker is

$$O = \{o_1, o_2, \ldots, o_K\} .$$

2. The set of alternative courses of action available to the decision maker is

$$A = \{a_1, a_2, \ldots, a_I\} .$$

3. The set of all possible future states which affect the utility of a particular alternative relative to a given objective is

$$\Theta = \{\theta_1, \theta_2, \ldots, \theta_J\} .$$

4. The evaluation of the utility of alternative $a_i$ with respect to objective $o_k$ when future $\theta_j$ is realized is

$$u_k(a_i,\theta_j) .$$

The sets A and $\Theta$ are indicated above to have a finite number of components. This is not required and either one or both could be denumerably infinite or even nondenumerably infinite. In the latter case, $u_k(\cdot,\cdot)$ would be in function form.

The objectives are assumed to be mutually exclusive to avoid double-counting the payoff. Some objectives will give rise to quantitative measures of their degree of attainment. For example, the objective "reduce scrap" would give rise to the measure "number of units scrapped." Other objectives are not naturally quantifiable. For example, the objective of obtaining a certain government contract has no scale of measurement associated with it. One might use the probability of obtaining the objective as a measure, but the point is that this is an artificially assigned quantity. In the following theory, it is assumed that $u_k$ can be given a value for each combination of a and $\theta$, although the units of measurement may vary from objective to objective. To derive a single quantity that can be used as a measure of effectiveness for an alternative when a given future occurs, it is necessary to sum the $u_k$ over all k (i.e., over all objectives). However, a standard unit of effectiveness must be chosen and all utility evaluations must be transformed into standard units before they can be added. The process is symbolized by

$$u(a_i,\theta_j) = \sum_k u_k^{(T)}(a_i,\theta_j), \text{ for } i = 1, 2, \ldots, I \text{ and } j = 1, 2, \ldots, J \,. \tag{4-1}$$

where $u(a_i,\theta_j)$ is the effectiveness of $a_i$ if $\theta_j$ occurs and $u_k^{(T)}(a_i,\theta_j)$ is $u_k(a_i,\theta_j)$ transformed into standard units.[1] This result is illustrated in Figure 4, where use of Equation (4-1) allows the structure (i) to be replaced by (ii).

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1. A standard unit of effectiveness used in engineering economy texts is discounted dollars. Here the transformation would involve, at least, application of present worth factors.

| | FUTURE $\theta_j$ | | | |
|---|---|---|---|---|
| | $0_1$ | $0_2$ | | $0_K$ |
| Alternative $a_i$ | $u_1(a_i,\theta_j)$ | $u_2(a_i,\theta_j)$ | . . . | $u_K(a_i,\theta_j)$ |

(i) Before Transformation and Summing.

| | FUTURE $\theta_j$ |
|---|---|
| Alternative $a_i$ | $u(a_i,\theta_j) = \sum_k u_k^{(T)}(a_i,\theta_j)$ |

(ii) After Transformation and Summing.

Figure 4. Computation of an Effectiveness Measure.

<u>Matrix Representation of a Decision Problem</u>. Once the effectiveness measure has been chosen and the computations indicated by Equation (4-1) carried out, the decision problem is of the following form:

| Action | FUTURE | | | |
|---|---|---|---|---|
| | $\theta_1$ | $\theta_2$ | | $\theta_J$ |
| $a_1$ | $u(a_1,\theta_1)$ | $u(a_1,\theta_2)$ | $\cdots$ | $u(a_1,\theta_J)$ |
| $a_2$ | $u(a_2,\theta_1)$ | $u(a_2,\theta_2)$ | $\cdots$ | $u(a_2,\theta_J)$ |
| . | . | . | | . |
| . | . | . | | . |
| . | . | . | | . |
| $a_I$ | $u(a_I,\theta_1)$ | $u(a_I,\theta_2)$ | $\cdots$ | $u(a_I,\theta_J)$ |

Figure 5. Matrix Representation of a Decision Problem.

Classification of Decision Problems. Decision problems may be classified relative to the degree of certainty with which futures can be predicted. To be specific, assume $\theta$ to be a random variable with probability distribution $P(\theta)$. Three classes of problems are commonly identified:

1. Decisions under certainty. This type of decision problem occurs when $P(\theta_j) = 1$, for some j. That is, some future is certain to occur.

2. Decisions under risk. This type occurs when $P(\theta)$ is known, but no particular future is certain.

3. Decisions under uncertainty. This type occurs when $P(\theta)$ is unknown. The uncertainty could come from a system of chance causes or from deliberate manipulation by a competitor.

It is recognized that few problems fit exactly into one of these categories. It would be rare if we knew $P(\theta)$ without error, but probably rarer yet if we knew nothing at all about $P(\theta)$. When $P(\theta)$ is estimated by a process involving the personal judgment of the decision maker, it is called a subjective probability distribution.

Principles of Choice. For each of the above types of problems, principles have been developed to select an alternative on the basis of the $u(a_i, \theta_j)$ and $P(\theta_j)$.

If $\theta_c$ is certain to occur, the course of action selected should be $a^*$, where

$$u(a^*, \theta_c) = \max_a u(a, \theta_c) \quad . \tag{4-2}$$

For decisions under risk several principles of choice are available:

1. Expected value principle. The course of action is chosen to maximize the expected utility; i.e., $a^*$ is selected such that

$$\begin{aligned} E[u(a^*, \theta)] &= \max_a E[u(a, \theta)] && (4\text{-}3) \\ &= \max_a \{\sum_\theta u(a, \theta)\ P(\theta)\} \\ &= \max_a U(a) \quad . \end{aligned}$$

Note that Equation (4-3) defines $U(a)$ as a symbol for the expected utility of action a.

2. Most probable future principle. The problem is treated as if under certainty, where $\theta_c$ in Equation (4-2) satisfies:

$$\theta_c = \max_{\theta_j} P(\theta_j) . \tag{4-4}$$

3. Expectation-Variance Principle. The course of action is to be selected on the basis of some function of the expected utility and the variance of the utility. The rule is to choose $a^*$ to satisfy

$$f\{U(a^*),V[u(a^*,\theta)]\} = \max_a f\{U(a),V[u(a,\theta)]\} , \tag{4-5}$$

where

$$V[u(a,\theta)] = \sum_{j=1}^{J} \{u(a,\theta_j) - E[u(a,\theta_j)]\}^2 P(\theta_j) . \tag{4-6}$$

4. Aspiration level principle. Instead of searching for all possible alternatives and applying one of the above criteria to find the "optimal" solution, the decision maker may be willing to accept the first alternative which meets a specified minimum level of performance. For example, the aspiration level (L) may be stated with respect to expected utility in the following manner: Choose the first alternative for which

$$U(a) \geq L . \tag{4-7}$$

Another use of the aspiration level concept is to constrain the selection of a course of action by a principle such as expected value.

If the decision maker desires to insure himself of a gain of L no matter what future was realized, he might state his principle of choice in the following manner: Choose $a^*$ which satisfies

$$U(a^*) = \max_{a \varepsilon A_1} U(a) , \tag{4-8}$$

where $A_1$ is a subset of the action set A and is defined by

$$A_1 = \{a_i : \min_{\theta} u(a_i, \theta) \geq L\} . \tag{4-9}$$

For decision problems under uncertainty the following principles of choice have been proposed:

1. Maximum principle. The principle of a pessimistic decision maker might be to choose the alternative which maximizes the worst possible gain that could result. The alternative selected is $a^*$ such that

$$\min_{\theta} u(a^*, \theta) = \max_{a} \min_{\theta} u(a, \theta) . \tag{4-10}$$

2. Maximax principle. This principle is as optimistic as the maximin is pessimistic. The procedure is to choose that alternative which maximizes the best possible gain that could result. Therefore $a^*$ is defined by

$$\max_{\theta} u(a^*, \theta) = \max_{a} \max_{\theta} u(a, \theta) . \tag{4-11}$$

(If a loss matrix, wherein losses were assigned positive values, were being analyzed, the role of "min" and "max" would be reversed in Equations (4-10) and (4-11), and the principles of choice would be called minimax and minimin, respectively.)

3. Hurwitz principle. In an effort to represent the philosophy of a decision maker who is neither ultra-pessimistic nor ultra-optimistic, one might consider the Hurwitz principle. This states that $a^*$ is chosen to maximize a weighted average of the best possible outcome and the worst possible outcome. For any alternative, a, the weighted average is written as

$$H(a;\alpha) = \alpha\{\max_{\theta} u(a,\theta)\} + (1-\alpha)\{\min_{\theta} u(a,\theta)\} \ . \tag{4-12}$$

The weight ($0 \leq \alpha \leq 1$) is called the "index of optimism." The action $a^*$ is chosen such that

$$H(a^*;\alpha) = \max_{a} H(a;\alpha) \ . \tag{4-13}$$

Note that for $\alpha = 1$, the maximax strategy is obtained, while for $\alpha = 0$, the maximin strategy results.

4. Minimax regret principle. The minimax regret principle was designed for decision makers who wish to avoid mental discomfort when analyzing their decisions in retrospect. The regret associated with the choice of action $a_i$ and the occurrence of future $\theta_j$ is defined as the difference between the value of the best decision for that future and what was actually realized; that is,

$$r(a_i,\theta_j) = \max_a u(a,\theta_j) - u(a_i,\theta_j) \ . \tag{4-14}$$

The minimax regret principle requires choice of an action $a^*$ which minimizes the maximum regret. Thus $a^*$ satisfies

$$\max_\theta r(a^*,\theta) = \min_a \max_\theta r(a,\theta) \ .$$

5. Laplace principle. Under this principle of choice, the action is selected on the basis of maximum expected value when all futures are considered equally likely. This application of Laplace's "Principle of Insufficient Reason" is based upon the philosophy that, if he is completely uncertain about $P(\theta)$, the decision maker cannot make rankings of the form $P(\theta_m) > P(\theta_n)$, thereby implying that he has no reason to doubt that $P(\theta_m) = P(\theta_n)$ for all m and n. If there are J possible futures and they are assumed equally likely, $a^*$ is chosen to maximize

$$U(a) = \frac{1}{J} \sum_{j=1}^{J} u(a,\theta_j) \ ,$$

which is the average of the entries in a row of the matrix of Figure 5.

The principles of choice listed above for decision problems under uncertainty might be applied over a subset of $\Theta$. For example, if the decision maker feels that a future in the subset $\Theta_1 \varepsilon \Theta$ is much more likely than a future in $\Theta - \Theta_1$, he may ignore all futures except those in $\Theta_1$. Yet he chooses to be uncertain of the probabilities associated with the elements of this subset.

To this point no mention has been made of the possibility of collecting data to aid in making decisions. Such data would be used to predict the future and hopefully would result in improved decisions. The objective of acceptance inspection is the obtaining of data upon which to base acceptance decisions. To allow evaluation of data collection procedures, it is necessary to use concepts from statistical decision theory.

## Statistical Decision Theory

Statistical decision theory was developed primarily by Abraham Wald (122) in order to describe and solve the following type of problem. A decision maker has a decision problem (described in the last section) and the possibility exists for collecting data upon which to base his decision. Among all possible data collection procedures (experiments) and all rules for action based upon the data collected, he must select the approach which is optimal with respect to decision losses and costs of experimentation. Note that the objective of statistical decision theory is to select an optimal decision process, which when carried out will assign the course of action to be adopted. Wald called the combination of experimental procedure and decision rule a "statistical decision function."

Components of a Statistical Decision Problem. The following are components of a statistical decision problem:

1. The set of all terminal actions available to the decision maker is $A = \{a_i\}$ .

2. The set of all possible futures (sometimes called states of nature) is $\Theta = \{\theta_j\}$ . The $\theta_j$ can be thought of as possible values of

a parameter, or parameters, in a probability distribution $f(x;\theta)$.

3. The effectiveness of action $a_i$ when the state of nature is $\theta_j$ is $u(a_i,\theta_j)$. It is assumed that all relevant objectives have been considered in arriving at this measure of utility. Further, it is supposed that $u(a_i,\theta_j)$ defines a preference order on the set A, such that for any $\theta_j$ the decision maker would prefer $a_n$ to $a_m$ if

$$u(a_n,\theta_j) > u(a_m,\theta_j) \ .$$

4. The set of possible outcomes of a given experimental procedure is $Z = \{z\}$ . The probability distribution of z is assumed to depend upon the parameter $\theta$.

5. The set of decision functions which specify the experimental procedure and for each z in Z assign an a in A is $D = \{d(z)\}$ .[2] For an observed z, $d(z) = a$. D is assumed to include the extreme cases of decision without experimentation.

6. A measure of utility (to the decision maker) of the decision function $d(z)$, when z is observed and $\theta$ is the true state of nature, is $u(z,d,\theta)$. This is the effectiveness described as Item 3, above, diminished by the cost of carrying out the experiment.

Relevant Probability Distributions. The following probability distributions are used in the analysis of statistical decision problems:

1. Prior distribution of $\theta$: $h(\theta)$. This distribution may be known or estimated by the analyst, or it may be assumed unknown.

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2. Only nonrandomized decision rules are considered.

2. Conditional probability of z, given θ: $f(z|\theta)$. This distribution is sometimes called the sampling distribution of z, or when evaluated for a particular z it may be referred to as the likelihood of the outcome.

3. Joint distribution of θ and z: $f(\theta,z)$. This can be calculated from

$$f(\theta,z) = f(z|\theta)\ h(\theta)\ . \tag{4-16}$$

4. Marginal distribution of z: $g(z)$. This distribution can be calculated from

$$g(z) = \sum_{\theta} f(\theta,z) \tag{4-17}$$

$$= \sum_{\theta} f(z|\theta)\ h(\theta)\ .$$

5. Posterior distribution of θ: $f(\theta|z)$. The conditional distribution of θ, having observed the outcome z, is given by

$$f(\theta|z) = \frac{f(\theta,z)}{g(z)}\ . \tag{4-18}$$

Equation (4-18) is a statement of Bayes Theorem. This fact is better seen from

$$f(\theta|z) = \frac{h(\theta)\ f(z|\theta)}{\sum_{\theta} h(\theta)\ f(z|\theta)}\ .$$

Thus the posterior distribution of $\theta$ is proportional to the product of the prior distribution and the likelihood of the observed outcome.

In the preceding definitions, no mention of the experimental procedure is apparent; however, this information is contained in the description of z.

Characteristics of Statistical Decision Functions. The performance characteristic of d(z) is the probability of selecting an action in some set $A_o \varepsilon A$, when $\theta$ is the true state of nature. It is given by

$$P_a(A_o|d,\theta) = \sum_{z:d(z)\varepsilon A_o} f(z|\theta) \ . \tag{4-19}$$

Note that $P_a$ is a function of $\theta$. When $A = \{a_1, a_2\}$ and $a_1$ is "acceptance" and $a_2$ is "rejection," $P_a(a_1|d,\theta)$ is called the operating characteristic function of d and $P_a(a_2|d,\theta)$ is called the power function.

The conditional expected utility with respect to z for given $\theta$ is called the utility characteristic of d for given $\theta$. It is defined by[3]

$$\bar{u}(d,\theta) = E_z[u(z,d,\theta)] \tag{4-20}$$

$$= \sum_z u(z,d,\theta) \ f(z|\theta) \ .$$

The utility characteristic is the expected payoff which would result from using decision function d when $\theta$ is the true state of nature. When

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3. Raiffa and Schlaifer (88, p. 13).

the utility function is replaced by a loss function, the expected loss is sometimes called the "risk."

Matrix Representation of a Statistical Decision Problem. The statistical decision problem is to select a decision function d(z), according to a principle of choice to be determined by the decision maker, when $\bar{u}(d,\theta)$ is given for all pairs $(d,\theta)$. The structure of this problem is shown in Figure 6. The form is similar to that of the no-data decision problem in Figure 5; however, now the problem is to choose an optimal decision rule rather than an optimal course of action. The no-data decision problem is contained within the structure of the general statistical decision problem.

| Decision Function | State of Nature | | | |
|---|---|---|---|---|
| | $\theta_1$ | $\theta_2$ | | $\theta_J$ |
| $d_1$ | $\bar{u}(d_1,\theta_1)$ | $\bar{u}(d_1,\theta_2)$ | . . . | $\bar{u}(d_1,\theta_J)$ |
| $d_2$ | $\bar{u}(d_2,\theta_1)$ | $\bar{u}(d_2,\theta_2)$ | . . . | $\bar{u}(d_2,\theta_J)$ |
| . | . | . | | . |
| . | . | . | | . |
| . | . | . | | . |
| $d_K$ | $\bar{u}(d_K,\theta_1)$ | $\bar{u}(d_K,\theta_2)$ | . . . | $\bar{u}(d_K,\theta_J)$ |

Figure 6. Matrix Representation of a Statistical Decision Problem.

Application of Principles of Choice. The principles of choice previously described with reference to the no-data decision problem can be applied to the matrix of Figure 6 to select a decision function. Only the expected value principle, the maximum principle, and the minimax regret principle will be discussed further.

The expected value principle states that d should be chosen to maximize the expected value over $\theta$ of the utility characteristic. Therefore the decision function $d^*(z)$ is optimal if

$$E_\theta[\bar{u}(d^*,\theta)] = \max_d E_\theta[\bar{u}(d,\theta)] \ . \qquad (4\text{-}21)$$

Use of this principle requires complete knowledge of the prior distribution of $\theta$, since

$$E_\theta[\bar{u}(d,\theta)] = \sum_\theta \bar{u}(d,\theta)\ h(\theta) \ . \qquad (4\text{-}22)$$

The decision function $d^*(z)$ is said to be "Bayes optimal" against the prior distribution $h(\theta)$.

By rearranging terms in Equation (4-21) it can be shown that the optimization does not have to be carried out in function space; rather, because of the special type of maximization process involved, it can be reduced to a number (exactly as many as there are points in Z) of simple minimizations over the set A. The following results start with the definition of $d^*(z)$:

$$E_\theta[\bar{u}(d^*,\theta)] = \max_d \ E_\theta[\bar{u}(d,\theta)] \tag{4-23}$$

$$= \max_d \ E_\theta \ E_z[u(z,d,\theta)]$$

$$= \max_d \sum_\theta \sum_z u(z,d,\theta) \ f(z|\theta) \ h(\theta)$$

$$= \max_d \sum_z g(z) \sum_\theta u(z,d,\theta) \ f(\theta|z)$$

$$= \max_d E_z \ E_{\theta|z} \ [u(z,d,\theta)] \ .$$

It is obvious that $d^*(z)$ has the property that for every z in Z, $d^*(z) = a_z^*$, where

$$E_{\theta|z}[u(z,a_z^*,\theta)] = \max_a E_{\theta|z}[u(z,a,\theta)] \ . \tag{4-24}$$

Therefore,

$$E_\theta[\bar{u}(d^*,\theta)] = \max_d E_z \max_a E_{\theta|z}[u(z,a,\theta)] \tag{4-25}$$

$$= \max_d \sum_z \{\max_a \sum_\theta u(z,a,\theta) \ f(\theta|z)\} \ g(z) \ .$$

The result stated by Equation (4-25) indicates that $d^*(z)$ can be constructed piecemeal by successively considering outcomes $z_1$, $z_2$, ..., and for each outcome defining $d^*(z)$ to be the course of action which maximizes the expected value of $u(z,a,\theta)$ with respect to the posterior distribution of $\theta$. The posterior distribution depends upon z and is defined by Equation (4-18).

The maximin principle is used when $h(\theta)$ is unknown and requires that $d^*(z)$ be chosen such that

$$\min_{\theta} \bar{u}(d^*,\theta) = \max_{d} \min_{\theta} \bar{u}(d,\theta) . \tag{4-26}$$

When the minimax regret principle is used, $d^*(z)$ is selected to satisfy

$$\max_{\theta} r(d^*,\theta) = \min_{d} \max_{\theta} r(d,\theta) \tag{4-27}$$

where the regret $r(d,\theta)$ is

$$r(d,\theta) = \max_{d} \bar{u}(d,\theta) - \bar{u}(d,\theta) . \tag{4-28}$$

Importance of Statistical Decision Theory. Statistical decision theory provides the methodology by which economic parameters, describing gains and losses, and statistical evidence, describing prior information about the process and current information from inspection, jointly can be considered to make inspection decisions. Relevant statistical considerations were described in Chapter III. The remainder of this chapter is devoted to cost considerations and to the selection of a principle of choice. Discussion of the application of statistical decision theory to acceptance inspection is contained in Chapters VI and VII.

## Gains and Losses Affected by Inspection Decision

### Difficulties of Definition

In analyses of industrial problems, where economic criteria are applicable to guide decision making, several terms are used to describe the influence of decisions on the economic well-being of the firm. Analysts invoke the terms "costs," "revenue," "losses," and "gains," often with no real attempt to define them. In fact, many times costs and losses are used interchangeably, as are revenue and gains. While it may seem desirable to have precise definitions for these terms, only general statements about their meaning in practice are possible.

One might define cost as the expenditure of resources incurred in carrying out a given act and revenue as the acquisition of resources resulting from the action. Both would be measured with respect to the state which would exist if the action were not taken and all other factors were held constant. Gains and losses are used in two general ways: (1) to represent the difference between revenue and cost, making gain (profit) and loss equal in absolute value but opposite in sign, and (2) as relative quantities, computed with reference to some standard of performance. In this latter usage, loss and opportunity cost would be synonymous.

In formal decision theory, a loss matrix often is formed instead of the matrix of Figure 5. The loss associated with a given alternative and future is denoted by $l(a,\theta)$ and could be defined generally as

$$l(a,\theta) = u(a_o^*,\theta_o) - u(a,\theta) \qquad (4\text{-}29)$$

where $\theta_o$ is a future selected as a datum for computing the loss and $a_o^*$ is the optimal action for that future. The equivalence of results from applying principles of choice to the loss matrix and those obtained previously for the matrix, heretofore called the "utility matrix," will be discussed later in this chapter. It may be convenient at this point to conceive of $\{u(a,\theta)\}$ as a "profit" matrix.

In the majority of reported analyses of acceptance inspection, emphasis has been on costs and losses affected by decisions. Any revenue has been treated as a negative cost. Three categories are considered: inspection costs, acceptance losses, and decision losses.

Inspection Costs

Investment Costs. A decision to establish an inspection operation at a point in the manufacturing process will usually require some expenditure for equipment and facilities and some increase in working capital.

Investment in equipment and facilities would be required for gauges, instruments, test racks, work benches, material handling equipment, tools, exhibits of inspection standards, and extension of plant services to the inspection station. Charges for the building space should not be made against the inspection operation, unless the existence of this operation required construction or rental of such space or created crowded conditions with resulting inefficiency of operation.

Additional working capital will be required to finance any additional in-process inventory created by the inspection operation. Lots of material waiting to be inspected, being inspected, or set aside for rectifying inspection constitute material whose time in process is lengthened because of the decision to inspect.

Operating Costs. The costs of operating an inspection station arise from several sources. Inspection labor, including supervision, is an expense. The costs of labor "extras," such as the employer's share of social security, workmen's compensation insurance, vacation relief, non-contributory insurance, and other benefits should be included. This labor cost is expended in such activities as sampling material, testing or measuring a product, analyzing inspection results, reporting inspection results, rewriting test specifications, handling material, and training.

Inspection supplies, plant services (power, water, fuel), product damaged or destroyed in inspection, equipment maintenance, and equipment rental are other classes of operating expense. Property taxes and insurance on inspection equipment are also included.

Treatment of Inspection Costs. The proper measure of the cost of an inspection operation depends upon the "life" of the activity. If the company is establishing this inspection merely for the duration of production of a given style of product, the proper time base of accumulating costs would be the time to produce the required quantity of product. However, if the inspection is to continue for an indefinite period of time (e.g., receiving inspection), costs must be put on some convenient time basis, such as a year.

The treatment of investment costs requires a charge to represent the opportunity cost incurred because resources are invested in the inspection activity at the expense of forfeiting the opportunity to invest in some other project. This charge is usually stated as the maximum annual rate of return that could be earned by investing in projects postponed by the decision to invest in the inspection operation. The usual time base for making this charge is annual.

By the same reasoning, expenditures which occur at future time points must be discounted at this rate of interest to reflect differences in the time utility of money.

Income taxes must be considered because they are an expense just as is labor cost. The appropriate tax rate depends upon federal, state, and local tax structures and the level of taxable income of the firm. The incremental tax rate should be applied against the change in taxable income attributable to inspection, in order to determine the change in income tax. Those investment items which are capitalized in the accounts of the firm are depreciated each year of their life. This depreciation serves to reduce taxable income and therefore income taxes.

Some costs, both investment and operating, depend upon the level of activity at the inspection operation. This level of activity is influenced by such factors as the rate of production, size of lots, sampling scheme, and test procedures. With increases in the production rate, more equipment and labor must be provided to keep pace. With large lots, the problem of obtaining a random sample becomes more difficult and the unit cost of sampling goes up; however, the frequency of taking samples will decrease. Large lots also require large amounts of floor space for storage. Sequential sampling plans may require repeated sampling from the lot, with a cost associated with each sampling stage. The size of the sample influences inspection costs, when materials or supplies are consumed in inspection, and when labor costs are dependent upon the number of units tested.

Some inspection costs are dependent upon the fraction defective of the product being inspected. The average sample size for sequential inspection procedures depends upon lot quality. Also the average total inspection per lot for single sampling rectifying schemes has been shown to be a function of incoming quality.

There are occasions where the cost of an inspection observation depends upon the value of the observation. This is true in some life tests and in cases where only defectives are destroyed in testing.

A model for placing inspection costs on an annual basis is given by Equations (4-30), (4-31), and (4-32). It is illustrated in Figure 7.

Let $I$ = initial investment required at time zero.

$C_k$ = estimated before-tax inspection cost in year k.

$L$ = projected duration of the inspection operation in years.

$t_k$ = estimated incremental tax rate in year k.

$C_k'$ = the portion of $C_k$ that is a deductible expense in determining tax liability in year k (not including depreciation charges).

$D_k$ = depreciation charges allowed in year k.

$i$ = minimum attractive rate of return after income taxes.

The after-tax cash flow in year k is estimated as[4]

$$C_k^* = C_k - t_k(C_k' + D_k) \ . \tag{4-30}$$

---

4. This assumes that the firm has taxable income in excess of $(C_k' + D_k)$ in year k. If the life for depreciation exceeds the project life, L is taken as the former.

The discounted cash flow is given by

$$\text{Discounted Cash Flow} = I + \sum_{k=1}^{L} C_k^*(1+i)^{-K} \ . \tag{4-31}$$

To convert this present worth to an equivalent uniform annual amount, the capital recovery factor (sometimes called the annuity factor) is applied, yielding

$$\text{Equivalent Uniform Annual Cost} =$$

$$\left[\frac{i(1+i)^L}{(1+i)^L - 1}\right]\left[I + \sum_{k=1}^{L} C_k^*(1+i)^{-K}\right] \ . \tag{4-32}$$

In case the duration of inspection activity is to be less than a year, no discounting is done and the total inspection costs for the period are obtained by summing the magnitude of the costs. A problem arises when equipment utilized at this operation can be used later in some other inspection activity. If future use is certain, prorating the capital recovery costs associated with ownership of the equipment is recommended.

Acceptance Losses

Internal Losses. Product that is accepted by inspection may result in economic losses to the producer. A defective item which is not detected in receiving inspection or at an in-process inspection will be processed through the following production operations, thereby having labor and machine time applied to it. The cost of this processing is

| Year | (1) Costs Before Taxes | Reduction in Taxable Income | | (2) Reduction in Income Taxes | (1)-(2) Net Cost After Taxes | Present Worth of After-Tax Cost |
|---|---|---|---|---|---|---|
| | | Depreciation | Other | | | |
| 0 | I | | | | I | I |
| 1 | $C_1$ | $D_1$ | $C_1'$ | $t(D_1+C_1')$ | $C_1^*$ | $C_1^*(1+i)^{-1}$ |
| 2 | $C_2$ | $D_2$ | $C_2'$ | $t(D_2+C_2')$ | $C_1^*$ | $C_2^*(1+i)^{-2}$ |
| . | . | . | . | . | . | . |
| . | . | . | . | . | . | . |
| . | . | . | . | . | . | . |
| L | $C_L$ | $D_L$ | $C_L'$ | $t(D_L+C_L')$ | $C_L^*$ | $C_L^*(1+i)^{-L}$ |
| | After-Tax Discounted Cash Flow = $I + \sum_{k=1}^{L} C_k^*(1+i)^{-k}$ | | | | | |

Figure 7. After-Tax Discounted Cash Flow for Inspection Costs.

avoidable, the loss being the sum of all resources expended on the item up to and including the operation wherein the defectiveness is finally discovered. It may be that a defective item could cause damage to equipment, other product, or personnel. To evaluate the loss associated with passing a defective item at an inspection at stage k in the process, the following approach could be used:

If $w_i$ is the probability a defective will be found at operation i, M is the number of operations, and $C_i$ is the expected loss when a defective item is processed from operation i through operation (i+1), the expected internal loss from passing a defective item at operation k is given by

$$L(k) = \sum_{i=k+1}^{M} \left\{ \left[ \sum_{j=k+1}^{i} C_j \right] \left[ w_i \prod_{j=k+1}^{i-1} (1-w_j) \right] \right\} . \qquad (4\text{-}33)$$

The above formulation considers only internal losses and does not not include the consequences of a defective reaching the consumer. The latter will be referred to as external losses.

External Losses. When a defective product reaches the consumer, the difficulties of measuring the economic effects are compounded. It may be that the item is technically defective in that it fails to conform to the established specifications, yet the consumer finds it satisfactory for his use. It may be that the customer accepts the defective item without complaint, yet he is unhappy with having received a substandard product. This discontent has been widely referred to as "loss of good will." One might conjecture that the loss of good will associated with a defective unit will not be constant but rather will be an increasing function of the number of defectives received before it. The effect of good will losses could be evidenced by the loss of sales in the same product or in other products bearing the company's name.

It may be that a customer will demand an adjustment on defectives that he discovers. He may return the item, requesting that an effective be sent in return or that his account be credited with the price of the item. He may use acceptance sampling in his receiving inspection, in which case an entire lot of items may be returned to the supplier. This act would result in the producer having to rehandle and possibly detail inspect good items as well as defectives. Loss of customer good will because a lot is rejected depends upon the customer's need for the material. If an arrangement existed where the customer sorted lots, rejected by his receiving inspection plan, at the producer's expense, the cost of such sorting would be an acceptance loss to the producer.

These external losses are extremely difficult to measure, although some firms have procedures for accumulating service and guarantee costs as well as data on reasons for return of product (82).

Analysts, in attempting to explicitly treat acceptance losses, have employed several forms of loss functions. The more common functions are listed below:

1. Losses associated with accepting a lot containing y defectives are proportional to y.

2. Losses associated with passing a lot having a fraction defective p are given by

$$L_a(p) = \begin{cases} 0, \text{ if } p \le p_o \\ C(p-p_o), \text{ if } p > p_o \end{cases} \tag{4-34}$$

where $p_o$ is a level of lot quality at which the losses from acceptance equal the losses from rejection.

3. Losses associated with accepting a product are proportional to the average outgoing quality.

Use of these models requires estimation of one or more cost parameters.

<u>Effect on Tax Liability</u>. It is estimated that most losses because

of acceptance of a defective product would be either an increase in operating costs or a decrease in sales revenue. Both would result in a reduction in taxable income and therefore in income tax liability. If t is the tax rate, the after-tax loss would be (1-t) times the before-tax loss.

Rejection Losses

Disposition of Rejected Product. The economic loss attributable to a rejected product depends upon the disposition of the material. Hald (53) illustrated some possible options with the diagram reproduced as Figure 8.

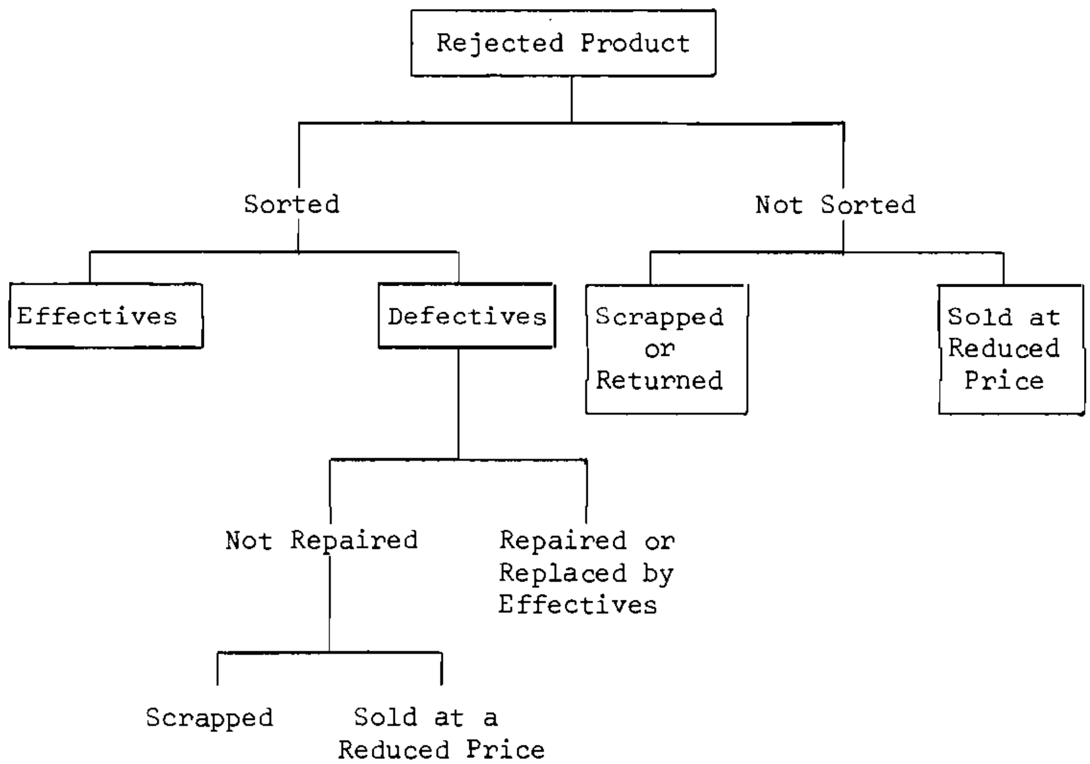


Figure 8. Disposition of Rejected Material.

Losses for Rejected Lots Which are Screened. If rejected lots are sorted, the cost of detail inspection of the remainder lot will be incurred. Also there is the cost of replacing or repairing defective items, which is to be offset by the value associated with the rectified item. Some revenue would be derived from the sale of defectives, if they are not rectified. If a given lot size is required by the consumer, either sufficient scrap allowance must be added to the size of the production run or else enough defectives must be rectified to meet the required lot size. This point illustrates the interaction between production economics and inspection economics.

Losses for Rejected Lots Which are not Screened. If rejected lots are not sorted, an obligation to produce a replacement lot may be incurred. This would result in production costs up to and including the inspection, with the risk that the new lot would be rejected, and so on. The unavailability of this material may cause production interruptions at future operations. If rejection occurs at receiving inspection and the lot is returned to the vendor, there may be losses which result from the delay in receiving a substitute shipment. Also, his added costs for handling and possibly sorting the rejected shipment may be reflected in the price of his product.

Two approaches have been taken to describe the effect of rejection losses when rejected lots are not screened.

1. Losses associated with rejecting a lot having a fraction defective p are given by

$$L_r(p) = \begin{cases} c(p_o - p), \text{ if } p \le p_o \\ 0, \text{ if } p > p_o \end{cases} \tag{4-35}$$

where $p_o$ is a level of lot quality at which the losses from acceptance and rejection are equal.

2. To use total cost per accepted lot, total cost per accepted item, or total cost per accepted effective item as the measure of effectiveness. Total cost includes the manufacturing cost of the item as well as inspection costs.

Tax Liability. As with acceptance losses, the after-tax effect of rejection losses would be approximately (1-t) times the before-tax value of the losses.

## Measure of Effectiveness

The analyst who wishes to formally consider economic factors has his choice between a decision model in which all relevant gains and losses are assumed measurable and a decision model in which some relevant gains and losses are assumed measurable, while others are treated by establishing aspiration levels on noneconomic variables.

In the former case, he may choose a "profit" model and symbolize the difference between revenues and costs as a function of the decision variables, or he may write an expression for the loss function. In the latter case, he expresses some of the economic factors in a profit function or a loss function and specifies certain conditions that the decision must satisfy in order to be admissible (e.g., a point on the OC-curve, a ceiling on the maximum possible loss). In both cases the measure of effectiveness will be in monetary terms.

It is recognized that this measure of effectiveness may not allow complete description of the decision maker's preference for alternatives. Many considerations are not able to be expressed in monetary terms and these must be evaluated subjectively, in association with quantitative results, in order to make decisions. However, it is submitted that monetary measures are the best available measure of the utility of a decision to the decision maker.

## Criteria for Choice

A decision maker expresses his personal philosophy when he selects a basis for choosing between alternatives. Principles of choice such as the Bayes, maximin, and minimax regret represent different outlooks about the future and different degrees of willingness to suffer the consequences of a poor decision.

The Bayes, or expected value, criterion presumes some knowledge of the prior distribution. Two analysts, working on the same problem, may assume different prior distributions and thereby recommend different decisions. Another apparent shortcoming of the Bayes principle is that it bases decisions on the average value and gives no consideration to extremes, other than as they affect the mean. Also, some decision makers hesitate to use decisions based on the "law of averages" for problems which occur only once. However, in spite of these criticisms, expected value is the principle of choice most often used in analysis of acceptance inspection.

To show that the minimization of expected loss yields the same preference relation between alternatives as does the maximization of utility, consider Equation (4-29) when alternative $a_1$ is preferred to alternative $a_2$:

$$E[l(a_1,\theta)] < E[l(a_2,\theta)]$$

$$E[u(a_o^*,\theta_o) - u(a_1,\theta)] < E[u(a_o^*,\theta_o) - u(a_2,\theta)]$$

$$u(a_o^*,\theta_o) - E[u(a_1,\theta)] < u(a_o^*,\theta_o) - E[u(a_2,\theta)]$$

$$E[u(a_1,\theta)] > E[u(a_2,\theta)] .$$

The last expression says that the expected utility of $a_1$ is greater than that of $a_2$, therefore $a_1$ is preferred to $a_2$.

The several principles of choice proposed for decision making under uncertainty have serious shortcomings, apart from the fact that one is almost never completely uncertain about future events.

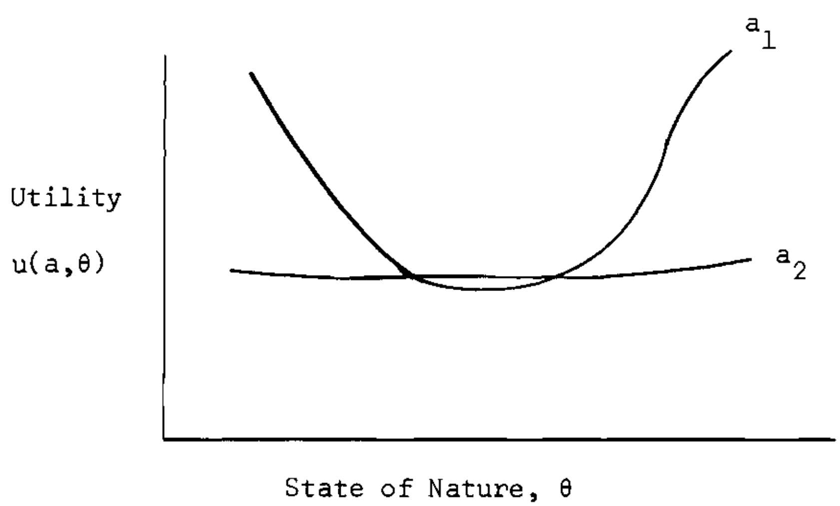


Figure 9. Illustration of a Shortcoming of the Maximin Criterion.

One criticism of the maximin utility principle is illustrated in Figure 9 (with $\theta$ shown as a continuous quantity). Although $a_1$ is superior to $a_2$ almost everywhere, application of the maximin principle would result in a choice of $a_2$. A second shortcoming of the minimax principle is its lack of the property of column linearity. This can be illustrated with the two utility matrices given below:

| | $\theta_1$ | $\theta_2$ | $\theta_3$ | min | |
|---|---|---|---|---|---|
| $a_1^*$ | 10 | 15 | 6 | 6 | max |
| $a_2$ | 8 | -6 | 5 | -6 | |
| $a_3$ | 3 | 12 | 18 | 3 | |

| | $\theta_1$ | $\theta_2$ | $\theta_3$ | min | |
|---|---|---|---|---|---|
| $a_1$ | 20 | 15 | 6 | 6 | |
| $a_2$ | 18 | -6 | 5 | -6 | |
| $a_3^*$ | 13 | 12 | 18 | 12 | max |

Analysis of the first matrix reveals that $a_1$ is maximin optimal. Suppose conditions change under $\theta_1$ such that the utilities for all alternatives are increased by ten units. The second matrix shows the result and indicates that $a_3$ is now minimax optimal, even though all alternatives were equally treated and the difference in utilities between alternatives remained constant.

The minimax regret principle suffers from a deficiency which is illustrated in the following example:

| | $\theta_1$ | $\theta_2$ | $\theta_3$ |
|---|---|---|---|
| $a_1$ | 11 | 15 | 7 |
| $a_2$ | 3 | 20 | 11 |
| $a_3$ | 14 | 8 | 10 |

(1a) Utility

| | $\theta_1$ | $\theta_2$ | $\theta_3$ | max | |
|---|---|---|---|---|---|
| $a_1^*$ | 8 | 7 | 0 | 8 | min |
| $a_2$ | 0 | 12 | 4 | 12 | |
| $a_3$ | 11 | 0 | 3 | 11 | |

(1b) Regret

| | $\theta_1$ | $\theta_2$ | $\theta_3$ |
|---|---|---|---|
| $a_1$ | 11 | 15 | 7 |
| $a_3$ | 14 | 8 | 10 |

(2a) Utility

| | $\theta_1$ | $\theta_2$ | $\theta_3$ | max | |
|---|---|---|---|---|---|
| $a_1$ | 0 | 7 | 0 | 7 | |
| $a_3^*$ | 3 | 0 | 3 | 3 | min |

(2b) Regret

For the first matrix, $a_1$ is optimal by the minimax regret principle when contrasted against $a_2$ and $a_3$. However, if $a_2$ is removed from consideration (second matrix) $a_3$ is preferred over $a_1$. This inconsistency is a serious shortcoming in the minimax regret principle.

Most decision problems in acceptance inspection are set in partial uncertainty about the prior distribution of lot quality. Application of principles of choice based on complete uncertainty is not warranted, especially in view of the many negative attributes of these principles. Through study of records, control chart data, vendor ratings, and subjective evaluation, the decision maker should be able to assign probabilities to the various possible lot qualities. (The mixed binomial is a convenient form for the prior distribution.) Application of the expected value principle, possibly constrained by some aspiration level, should provide a rational basis for decision.

## Economic Implications of Commonly Used Acceptance Sampling Procedures

### Noneconomic Measures of Effectiveness

The foregoing discussion of statistical decision theory indicates that a formal structure is available for analysis of acceptance sampling decisions. However, most currently used sampling procedures were not constructed through consideration of loss functions and prior distributions. The designers used related but nevertheless noneconomic criteria as the basis for their tables. Two quantities, the operating characteristic curve and the average outgoing quality curve, provide crude measures of acceptance losses and the OC-curve also gives the risk of rejecting good material. The average sample number is a measure of inspection cost, while the average total inspection per lot is a measure of inspection cost and rejection losses because of sorting. All of these measures are functions of process quality and therefore require some information about the process for their evaluation.

Dodge-Romig Tables

In Chapter II the criteria by which the Dodge-Romig plans were derived are described. Under the AOQL procedures, it evidently is assumed that acceptance losses are proportional to the average outgoing quality and that inspection costs, which include the rejection loss because of screening, are proportional to the number of units inspected. The principle of choice is to minimize the average amount of inspection per lot, subject to a limit on the maximum average outgoing quality. Thus with simple loss functions, the average losses due to inspection of the sample and rejected lots are to be minimized (Bayes principle) subject to control of acceptance losses by stating the AOQL (aspiration level). The LTPD is similar in nature, differing only in that the aspiration level is the consumer's risk point.

The prior distribution is roughly a two-point binomial: one process fraction defective called "normal" and a second level of process quality considered unsatisfactory.

It should be emphasized that Dodge and Romig developed their plans without formal use of loss functions or prior distributions.

Military Standard 105D

The economic implications underlying MIL-STD-105D are more obscure than those of the Dodge-Romig plans. The fact that the sample size increases with lot size and inspection level reflects the larger economic losses associated with wrong decisions about the lots. Further, the probability of accepting a lot having a fraction defective equal to the AQL exceeds 0.90 for most plans in the tables. Thus the risk of rejecting acceptable lots is constrained. The plan makes use of sample data to estimate the process average and employs the estimate to determine whether tightened or reduced inspection is to be implemented. The sample size and therefore sampling costs are reduced appreciably if the producer can qualify for reduced inspection.

## Summary

Initially presented in this chapter was decision theory, which describes the manner in which a decision maker jointly considers objectives, alternative courses of action, and various possible future circumstances in order to select the alternative which is optimal according to his measure of utility and principle of choice. Principles of choice were presented for decision making under certainty, risk, and uncertainty. The framework was extended to encompass statistical decisions, wherein information obtained from experimentation is utilized in making the decision. It was pointed out that statistical decision theory integrates statistical considerations, such as those presented in

Chapter III, with economic considerations.

Gains and losses influenced by acceptance inspection decisions were described under three categories: inspection costs, acceptance losses, and rejection losses. A model was proposed for finding the after-tax annual cost of inspection. A method for computing the internal losses caused by passing defective material was presented. It was shown that the losses associated with rejection of the lot depend upon the disposition of the material, and several possibilities were discussed. The effect of income taxes upon these classes of gains and losses was described.

It was concluded that monetary units represent the best available measure of effectiveness for inspection decisions. Examination of the principles of choice led to the conclusion that the Bayes principle, perhaps constrained by specification of aspiration levels, is the logical criterion for acceptance inspection decisions. It was shown that an alternative which is Bayes optimal with respect to a loss matrix is also Bayes optimal with respect to the corresponding utility matrix.

Finally, the noneconomic criteria used in designing commonly used inspection procedures were discussed. The economic implications underlying the Dodge-Romig Tables and Military Standard 105D were mentioned briefly.

# CHAPTER V

# ADDITIONAL CONSIDERATIONS IN DESIGN OF ACCEPTANCE INSPECTION SYSTEMS

## General

The purpose of this chapter is to discuss briefly some additional considerations which affect the design of inspection systems. In general they are concerned with the human factor in inspection. In particular, such topics as the accuracy of inspection, the effect of learning in inspection, the psychological effects of the presence of inspection, possible competition between producer and consumer, and certain organizational considerations are treated.

## Accuracy of Inspection

Whether done manually, automatically, or by a combination of man and instrument, errors in measuring and classifying a product will occur. Human causes may be deliberate as well as unintentional. The consequences of inspection errors might be a rejected good product, an accepted defective product, or unnecessary sampling under a sequential scheme.

### Human Error

Human error may result from carelessness, failure to understand specifications, inability to correctly carry out the inspection method, or hesitation to classify defective an item which could cause an entire lot to be rejected. These errors may be a function of monotony or fatigue. Thus there is the popular concept that accuracy of inspection is greater when sampling than when screening. The influence of fatigue should be a function of the number of units inspected during the shift, the nature and time of work breaks, and the number of units which the inspector believes remain to be inspected. Monotony is reduced by varying inspection tasks; even the occurrence of a defective is a change of activity for the inspector.

Incorporation of the influence of human errors into models for decision making requires the ability to formulate the probabilities of various kinds of inspection errors as functions of the decision variables and process parameters. The following simple example serves to illustrate the approach.

Example. An inspector manually inspects a part in lot-by-lot attribute single sampling. There is no instrument error. Assume that the only possible type of inspection error is the classification of a defective as acceptable. Effective items are correctly identified. Assume that the probability of misclassifying a defective is constant throughout inspection and is a monotone increasing function of the sample size. Denote this probability by $e(n)$ and let $r$ be the number of defectives classified as effective. Then the probability distribution of $r$, given that the sample contained $x$ defectives, would be

$$f_n(r|x) = \binom{x}{r} [e(n)]^{r} [1 - e(n)]^{x-r}, \quad r = 0, 1, \ldots, x. \qquad (5\text{-}1)$$

The probability of rejecting a lot is the probability that $x-r$ will exceed the acceptance number $c$:

$$P(x-r > c) = \sum_{x=c+1}^{n} \sum_{r=0}^{x-c-1} f_n(r|x)\ g_n(x) \ ,$$

where $g_n(x)$ depends upon the prior distribution of lot quality. Equation (5-2) may be written as

$$P(x-r > c) = \sum_{x=c+1}^{n} \sum_{r=0}^{x-c-1} \binom{x}{r} [e(n)]^{r} [1 - e(n)]^{x-r} \ g_n(x)$$

$$= \sum_{x=c+1}^{n} g_n(x) - \sum_{x=c+1}^{n} g_n(x)\ F(e(n);x-c-1,c) \qquad (5\text{-}3)$$

where the first term is the probability of acceptance when no inspection errors are possible and the second term involves the incomplete beta function,

$$F(e(n);x-c-1,c) = \frac{x!}{c!(x-c-1)!} \int_{0}^{e(n)} t^{x-c-1}(1-t)^{c}dt \ . \qquad (5\text{-}4)$$

Because the function of Equation (5-4) is non-negative, it is seen that the probability of acceptance has been reduced as a result of the possibility of inspection error. This analysis could be extended to modify the statistical properties given for attribute sampling plans in Chapter III.

Instrument Error

Errors introduced by instruments which measure quality characteristics are a function of the accuracy and precision of the measuring device. Accuracy can be controlled only through periodic calibration, while precision can be improved through replication of the measuring process.

For example, suppose the quality characteristic x is normally distributed with mean $\mu$ and variance $\sigma^2$. The specification limits are (L,U). The true probability that an item is effective is

$$p_T = \int_L^U N(\mu,\sigma^2)\ dx \tag{5-5}$$

where

$$N(\mu,\sigma^2) \equiv \left[\frac{1}{\sigma\ \sqrt{2\pi}}\right] e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} . \tag{5-6}$$

If the measuring instrument has normally distributed error with mean B and variance $\sigma_e^2$, the proportion of items classified effective on the basis of a single measurement, m, on each item is[1]

$$p_M(1) = \int_L^U N(\mu+B,\ \sigma^2 + \sigma_e^2)\ dm . \tag{5-7}$$

1. The measured value m equals x plus the error e; therefore $E(m) = E(x) + E(e)$ and $V(m) = V(m) = V(x) + V(e)$, if e and x are independent.

The probability that an effective item, having $L \le x \le U$, is classified correctly can be calculated from

$$P(L \le m \le U | L \le x \le U) = \frac{P(L \le m \le U,\ L \le x \le U)}{P(L \le x \le U)} . \tag{5-8}$$

The numerator of Equation (5-8) is given by

$$P(L \le m \le U,\ L \le x \le U) = \int_L^U \int_L^U f(m,x)\ dm\ dx \tag{5-9}$$

$$= \int_L^U \int_L^U N(\mu + B,\ \sigma^2 + \sigma_e^2)\ N(\mu, \sigma^2)\ dm\ dx\ .$$

The probability that an effective item is classified incorrectly is one minus the value obtained from Equation (5-8).

The probability that a defective item, with $x < L$ or $x > U$, will be classified correctly is given by

$$P(m < L | x < L) + P(m > U | x > U) + P(m < L | x > U) + P(m > U | x < L) \tag{5-10}$$

$$= \frac{P(m < L,\ x < L)}{P(x < L)} + \frac{P(m > U,\ x > U)}{P(x > U)} + \frac{P(m < L,\ x > U)}{P(x > U)} + \frac{P(m > U | x < L)}{P(x < L)} .$$

The probability of incorrectly classifying a defective is one minus the result given by Equation (5-10).

The probabilities of correct decisions, given by Equations (5-8) and (5-10), can be increased by making replicate measurements and using the average as the basis for classifying the item. If k replicate measurements are made and if $\bar{m}$ is their mean, the proportion of items classified effective will be

$$p_M(k) = \int_L^U N(\mu+B,\ \sigma^2 + \sigma_e^2/k)\ d\bar{m} . \qquad (5\text{-}11)$$

Equations (5-8), (5-9), and (5-10) must be modified by replacing m and its distribution with $\bar{m}$ and associated distribution.

To determine the number of replicates under detail inspection, let

A = act of accepting a defective through error,

R = act of rejecting an effective through error,

$L_A$ = loss associated with accepting a defective,

$L_R$ = loss associated with rejecting an effective, and

C = cost of a measurement.

The desired number of replicates k* minimizes

$$C\ k + L_A\ P(A) + L_R\ P(R) \qquad (5\text{-}12)$$

where the probabilities, P(A) and P(R), are obtained from the previous results.

## Influence of Learning

The time required to inspect the stipulated characteristics of an item should decrease as the inspector gains experience in carrying out the procedures and commits the specifications to memory. Errors of judgment should also decline with time. Typically these declines will taper off to a plateau when learning is complete. If the amount of inspection is large compared to that required in the learning process, the effect of learning on inspection costs may be negligible; otherwise explicit description of the learning process will be required to properly describe the economic consequences of the inspection acts.

## Psychological Effects of Inspection

In Chapter II, the possibility of the chosen inspection policy affecting quality at prior operations was mentioned. It was stated that many analysts believe that inspection, in conjunction with a penalty for poor quality, can result in a significant improvement in quality. If this possibility exists, the preventive effect will have to be evaluated in terms of decision variables and induced changes in the prior distribution.

The significance of any such effect is that it means the prior distribution will depend upon the decision; or to use the terminology of decision theory, the probability measure defined on the set of possible futures is conditional on the decision function adopted. Thus the amount of information needed to analyze the decision problem has been increased greatly. Also, theories and results of dynamic programming cannot be utilized, because action at a given inspection station affects both prior and future performance.

A vendor may be influenced by knowledge that the buyer will inspect lots that he is sent. In an effort to improve his quality, the vendor may introduce inspection and other control procedures which may eventually necessitate a price increase. The vendor also may exploit the weaknesses in an acceptance sampling plan used by the buyer. Resubmission without rectification of a lot previously rejected by the buyer is an example.

Morale of vendors or production units within the organization may be influenced by inspection decisions. The prestige of allowing a production operation to operate under "reduced" inspection may be valued by workers; while having to rectify a rejected lot while off incentive pay may demoralize workers. Vendors that the firm wants to cultivate as reliable sources of supply should not be alienated by offhand rejection of product.

The fact that a firm has a well conceived inspection program may impress potential customers. Many organizations advertise their quality control program in order to establish buyer confidence. It is common for governmental agencies to require adequate inspection procedures before they will grant contracts.

## Competition Between Producer and Consumer

Ignoring the moral implications, consider the decisions a producer makes regarding finished product which he will send to a customer. The producer's decision about final inspection well may be influenced by what he thinks the consumer will do about receiving inspection. Conversely, the consumer's actions may be guided by his evaluations of the producer's intentions. Thus the producer suffers from uncertainty about the economic consequences of material sent to his customer, while the customer suffers from uncertainty about the quality of material which he will receive--the uncertainties arising partly from chance and partly from deliberate action by the "adversary." While it may be natural to assume that producer and consumer would cooperate to their mutual advantage, other circumstances may require consideration of the competitive aspects of the producer-consumer relationship.

## Organizational Considerations

The analysis of inspection systems should not ignore the influence of organization and administrative policies. For example, if acceptance inspection is administratively separate from the production function, compromises on accepting substandard material in the interest of meeting production goals are not made without proper consideration by a higher management level. Further, the method of allocation of inspection costs between the inspection department and production departments could serve as an incentive to improve quality. If production were charged with all rectifying inspection costs or else had to furnish the labor for such tasks, there would be some reward to production management if they improved quality.

## Summary

Several considerations, additional to those of a statistical or economic nature, were discussed in this chapter. Methods for quantifying the effect of inspection errors by instruments and inspectors were given. A procedure was described for determining the economically optimal number of replicate measurements to make on an item. Brief qualitative statements were made about the influence of learning on inspection costs, the psychological effects of inspection, the possibility of competition between producer and consumer, and organizational considerations. No attempt was made to formally analyze these latter considerations, because it was felt that no general theory could be developed within the scope of this research.

# CHAPTER VI

# METHODOLOGY FOR SYSTEM DESIGN

## General

The purpose of this chapter is to provide a conceptual model of an acceptance inspection system and to discuss methods of approach to the design of system components. The methodology is illustrated through application of statistical decision theory to design acceptance procedures in a single-stage inspection system. Models are developed for rectifying and nonrectifying systems. Illustration of the design of a multistage system is deferred to the next chapter.

## Conceptual Model of an Inspection System

### An Interrelated Sequence of Operations

A graphic representation of a production-inspection system is shown in Figure 10. Material acquired from vendors is subject to a possible receiving inspection and thereafter the product may be inspected for acceptability following each production operation. The squares, which represent acceptance inspection operations, are drawn with dashed lines to indicate that their existence is yet to be determined. Conceivably, the firm could operate with no acceptance inspection. Which of the possible inspection locations to activate and the nature of the operations at each station are problems of the systems designer. He would like to manipulate the variables under his control to maximize some overall measure of system effectiveness. Unless the individual inspection stations operate independently, system components must be specified with consideration of their effects on other components. The nature of these dependencies is exposed to some extent by a study of a single inspection operation.

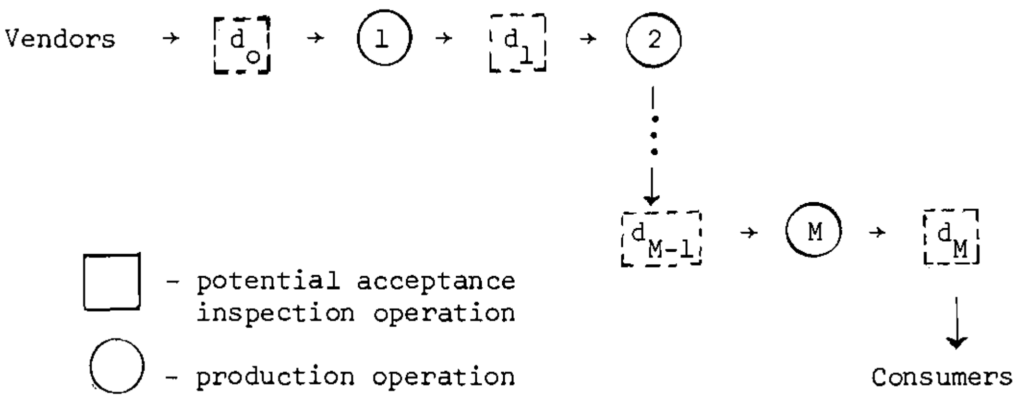


Figure 10. Representation of a Multistage Inspection System.

A Single Inspection Operation

Figure 11 depicts a single inspection operation which receives material whose quality is described by the parameter set $\theta$ and its probability measure $h(\theta)$. A decision function $d(z)$ is applied to determine the disposition of the material. The result is material leaving the inspection operation having a quality level described by $h'(\theta)$. Quality is not the only product characteristic affected by inspection. Certain types of inspection procedures may affect the quantity of material which flows. If the value of the product is thought to be dependent upon its quality and quantity, then inspection could be conceived as an operation inserted between two production activities for the purpose of improving the value of the product flow. This increase in value must offset the cost of the inspection operation itself. The inspection cost is dependent upon the quality and quantity received as well as the decision function d(z).

| Input | | | Acceptance Inspection | | Output | |
|---|---|---|---|---|---|---|
| Quality: | $h(\theta)$ | $\rightarrow$ | Decision Function $d(z)$ | $\rightarrow$ | Quality: | $h'(\theta)$ |
| Quantity: | $Q$ | | | | Quantity: | $Q'$ |
| Value: | $V(h,Q)$ | | | | Value: | $V(h',Q')$ |

Figure 11. Graphical Representation of a Single Inspection Operation.

The choice of a decision function d (here assumed to represent a complete specification of the design characteristics of the inspection operation) will influence the output going to future processes and therefore the input to any later inspection operations. By this same reasoning, the input to the inspection operation in question was affected by decision functions at prior inspection stations. To complicate matters further, the decision function adopted at this inspection station may influence prior production (and perhaps inspection) operations so that the input is modified. This means that if one chooses a decision function based on a given set of input characteristics and carries out inspection accordingly, the input characteristics may change, making the chosen function no longer optimal.

System Effectiveness

Very generally it may be stated that the objective of inspection system design is to choose optimal decision functions $d_o^*$, $d_1^*$, ..., $d_M^*$ such that the value added by inspection less the cost of inspection is maximized.

To do this formally would require the ability to formulate an expression for system effectiveness in terms of controllable variables, estimate values for the relevant parameters, and mathematically treat the formulation to obtain a solution, optimal with respect to a principle of choice which expresses the decision maker's philosophy about uncertainty created by the uncontrollable variables. All of these tasks are difficult.

Computational Problems

Even if the model of system effectiveness could be constructed and the parameters reliably estimated, the difficulties of searching for the optimal design might be insurmountable. For example, the restricted design problem of determining whether or not to detail inspect product at one or more of eight possible inspection points requires consideration of $2^8$, or 256, different designs. (Under certain helpful assumptions about the interdependencies between operations, this particular problem has been solved by Lindsey and Bishop (79).)

Dynamic programming appears to offer promise as an optimization technique in specialized situations where the dimensionality of the problem is not large. Systematic search methods starting from a design based on experience and judgment may lead to an improved, if not optimal, design. Simulation techniques may be utilized to test proposed designs against assumed process conditions. The sensitivity of the effectiveness measure to design changes or errors in estimating parameters also can be evaluated by this technique.

In the case of a single inspection point, the optimization process possibly could be carried out by conventional calculus methods. Two examples which illustrate the method of designing an attribute sampling plan follow. Methodology for multistage optimization is given in the next chapter.

## Design of a Single-Stage Inspection System

### Application of Statistical Decision Theory to Acceptance Sampling

In Chapter IV, the principles of decision theory were outlined and stated to be fundamental to the analysis of acceptance inspection. Before presenting the complex utility functions of the examples which follow, it is desirable to study a simple illustration of statistical decision theory applied to determine the optimal decision function for attribute inspection.

Assume a lot of five items is presented for acceptance. At most two items may be inspected. The cost of inspecting an item is \$1.00, the loss if a defective item is passed is \$5.00, and the loss if a lot is rejected (to be rectified) is \$2.00 for every item in the remainder lot. This loss model is not realistic; its only true virtue is expediency. Emphasis here is on the method of computation of the optimal decision function, rather than the construction of a utility function.

The components of the decision problem are as follows:

1. Alternative courses of action: $A = \{a_1, a_2\}$, where

$a_1$ = accept the lot.

$a_2$ = reject the lot.

2. States of nature: $\Theta = \{X{:}X = 0, 1, 2, \ldots, 5\}$, where $X$ = number of defectives in the lot.

3. Decision functions: $D = \{d(z)\}$, where the experiment is to randomly sample and inspect n units for defectives.

4. Outcomes of the experiment: $Z = \{x{:}x = 0, 1, \ldots, n\}$, where x is the number of defectives in the sample. (The order in which the defectives occur yields no information if random sampling is employed.)

5. Loss function: $l(x,d,X)$, the loss if a lot containing X defectives is inspected according to a decision function d and x defectives occur in the sample. For this example

$$l(x,d,X) = \begin{cases} n + 5(X-x) \text{ , if } d(x) = a_1 \\ \\ n + 2(5-n) \text{ , if } d(x) = a_2 \end{cases} \qquad (6\text{-}1)$$

The 14 possible decision functions are listed in Table 1.

Table 1. List of Possible Decision Functions, Showing Action Taken for Each Outcome

| Decision Function | Sample Size | Outcome (x) 0 | 1 | 2 |
|---|---|---|---|---|
| $d_a$ | 0 | (Accept Without Experimentation) | | |
| $d_r$ | 0 | (Reject Without Experimentation) | | |
| $d_1$ | 1 | $a_1$ | $a_1$ | |

Table 1. List of Possible Decision Functions, Showing Action Taken for Each Outcome (Continued)

| Decision Function | Sample Size | Outcome (x) 0 | 1 | 2 |
|---|---|---|---|---|
| $d_2$ | 1 | $a_1$ | $a_2$ | |
| $d_3$ | 1 | $a_2$ | $a_1$ | |
| $d_4$ | 1 | $a_2$ | $a_2$ | |
| $d_5$ | 2 | $a_1$ | $a_1$ | $a_1$ |
| $d_6$ | 2 | $a_1$ | $a_1$ | $a_2$ |
| $d_7$ | 2 | $a_1$ | $a_2$ | $a_1$ |
| $d_8$ | 2 | $a_1$ | $a_2$ | $a_2$ |
| $d_9$ | 2 | $a_2$ | $a_1$ | $a_1$ |
| $d_{10}$ | 2 | $a_2$ | $a_1$ | $a_2$ |
| $d_{11}$ | 2 | $a_2$ | $a_2$ | $a_1$ |
| $d_{12}$ | 2 | $a_2$ | $a_2$ | $a_2$ |

Values for the loss function, $l(x,d,X)$, are given in Table 2. The probability distribution $f(x|X)$ is shown in Table 3 and values for the expected loss

$$\bar{l}(d,X) = \sum_{x} l(x,d,X)\ f(x|X) \qquad (6\text{-}2)$$

are given in Table 4.

To obtain the expected loss over all states of nature, the prior distribution of X must be stated and used in the following:

$$L(d) = \sum_{X} \bar{l}(d,X)\; h(X) \tag{6-3}$$

For h(X) given in Table 5, L(d) has been computed for each d and entered in Table 6. Examination of Table 6 reveals $d_6$ to be the decision function which minimizes total expected losses. Because of the unrealistic loss function no further comment is made regarding this example. The objective of illustrating the method for applying statistical decision theory in the selection of acceptance sampling procedures has been accomplished. Two examples follow in which the emphasis is on the construction of the utility function. The first is concerned with rectifying inspection and the second with nonrectifying inspection.

Selection of an Attribute Single Sample Plan for Rectifying Inspection

An attribute single sample plan (n,c) is to be selected for lot-by-lot inspection. It is assumed that there are no errors in inspection and the prior distribution is independent of the plan selected. The following notation will be used:

$N$ = lot size

$n$ = sample size, $0 \le n \le N$

$c$ = acceptance number, $c = 0, 1, \ldots, n$

$X$ = number of defectives in the lot

$x$ = number of defectives in the sample

$\bar{p} = E(X)/N$

$y = X - x$, the number of defectives in the remainder lot

$S_F$ = fixed cost per lot if any inspection is carried out

Table 2. Values of the Loss Function, l(x,d,X)

| Decision Function | Sample Size | NUMBER OF DEFECTIVES IN THE LOT (X) 0 | 1 | | 2 | | | 3 | | | 4 | | | 5 | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | x=0 | x=0 | x=1 | x=0 | x=1 | x=2 | x=0 | x=1 | x=2 | x=0 | x=1 | x=2 | x=0 | x=1 | x=2 |
| $d_a$ | 0 | (0) | (5) | | (10) | | | (15) | | | (20) | | | (25) | | |
| $d_r$ | 0 | (10) | (10) | | (10) | | | (10) | | | (10) | | | (10) | | |
| $d_1$ | 1 | 1 | 6 | 1 | 11 | 6 | * | 16 | 11 | * | 21 | 16 | * | ** | 21 | * |
| $d_2$ | 1 | 1 | 6 | 9 | 11 | 9 | * | 16 | 9 | * | 21 | 9 | * | ** | 9 | * |
| $d_3$ | 1 | 9 | 9 | 1 | 9 | 6 | * | 9 | 11 | * | 9 | 16 | * | ** | 21 | * |
| $d_4$ | 1 | 9 | 9 | 9 | 9 | 9 | * | 9 | 9 | * | 9 | 9 | * | ** | 9 | * |
| $d_5$ | 2 | 2 | 7 | 2 | 12 | 7 | 2 | 17 | 12 | 7 | ** | 17 | 12 | ** | ** | 17 |
| $d_6$ | 2 | 2 | 7 | 2 | 12 | 7 | 8 | 17 | 12 | 8 | ** | 17 | 8 | ** | ** | 8 |
| $d_7$ | 2 | 2 | 7 | 8 | 12 | 8 | 2 | 17 | 8 | 7 | ** | 8 | 12 | ** | ** | 17 |
| $d_8$ | 2 | 2 | 7 | 8 | 12 | 8 | 8 | 17 | 8 | 8 | ** | 8 | 8 | ** | ** | 8 |
| $d_9$ | 2 | 8 | 8 | 2 | 8 | 7 | 2 | 8 | 12 | 7 | ** | 17 | 12 | ** | ** | 17 |
| $d_{10}$ | 2 | 8 | 8 | 2 | 8 | 7 | 8 | 8 | 12 | 8 | ** | 17 | 8 | ** | ** | 8 |
| $d_{11}$ | 2 | 8 | 8 | 8 | 8 | 8 | 2 | 8 | 8 | 7 | ** | 8 | 12 | ** | ** | 17 |
| $d_{12}$ | 2 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | 8 | ** | 8 | 8 | ** | ** | 8 |

* Sample size less than x.   ** This few defectives impossible.

Table 3. Probability Distribution of x, Given X, When N = 5

| Sample Size | Outcome x | X=0 | X=1 | X=2 | X=3 | X=4 | X=5 |
|---|---|---|---|---|---|---|---|
| | 0 | 1 | .6 | .3 | .1 | 0 | 0 |
| 2 | 1 | 0 | .4 | .6 | .6 | .4 | 0 |
| | 2 | 0 | 0 | .1 | .3 | .6 | 1 |
| 1 | 0 | 1 | .8 | .6 | .4 | .2 | 0 |
| | 1 | 0 | .2 | .4 | .6 | .8 | 1 |

Table 4. Values of the Loss Characteristic, $\bar{l}(d,X)$

| Decision Function | NUMBER OF DEFECTIVES IN THE LOT | | | | | |
|---|---|---|---|---|---|---|
| | 0 | 1 | 2 | 3 | 4 | 5 |
| $d_a$ | 0 | 5.0 | 10.0 | 15.0 | 20.0 | 25.0 |
| $d_r$ | 10.0 | 10.0 | 10.0 | 10.0 | 10.0 | 10.0 |
| $d_1$ | 1.0 | 5.0 | 10.0 | 15.0 | 20.0 | 21.0 |
| $d_2$ | 1.0 | 6.4 | 10.6 | 14.6 | 18.6 | 9.0 |
| $d_3$ | 9.0 | 7.4 | 8.4 | 9.4 | 10.4 | 21.0 |
| $d_4$ | 9.0 | 9.0 | 9.0 | 9.0 | 9.0 | 9.0 |
| $d_5$ | 2.0 | 5.0 | 8.0 | 11.0 | 14.0 | 17.0 |
| $d_6$ | 2.0 | 5.0 | 8.6 | 11.3 | 11.6 | 8.0 |
| $d_7$ | 2.0 | 7.4 | 8.6 | 8.6 | 10.4 | 17.0 |
| $d_8$ | 2.0 | 7.4 | 9.2 | 8.9 | 8.0 | 8.0 |
| $d_9$ | 8.0 | 5.6 | 6.8 | 9.1 | 14.0 | 17.0 |
| $d_{10}$ | 8.0 | 5.6 | 7.4 | 10.4 | 11.6 | 8.0 |
| $d_{11}$ | 8.0 | 8.0 | 7.4 | 7.7 | 10.4 | 17.0 |
| $d_{12}$ | 8.0 | 8.0 | 8.0 | 8.0 | 8.0 | 8.0 |

Table 5. Prior Distribution of Lot Quality, h(X)

| X | h(X) |
|---|---|
| 0 | .3 |
| 1 | .4 |
| 2 | .1 |
| 3 | .1 |
| 4 | .1 |
| $\geq$ 5 | 0 |

Table 6. Bayes Loss Function, L(d)

| Decision Function | L(d) | |
|---|---|---|
| $d_a$ | 6.50 | |
| $d_r$ | 10.00 | |
| $d_1$ | 6.80 | |
| $d_2$ | 7.24 | |
| $d_3$ | 8.48 | |
| $d_4$ | 9.00 | |
| $d_5$ | 5.90 | |
| $d_6$ | 5.75 | Minimum |
| $d_7$ | 6.32 | |
| $d_8$ | 6.17 | |
| $d_9$ | 7.63 | |
| $d_{10}$ | 7.58 | |
| $d_{11}$ | 8.15 | |
| $d_{12}$ | 8.00 | |

$s_2$ = unit cost of inspecting and classifying

$s_1$ = unit cost of sampling

$C_M$ = cost of reproducing an item presented for inspection (the "value" of an item received)

$V_1$ = value associated with an accepted effective item

$V_2$ = value associated with an accepted defective item

$C_r$ = cost of repairing a defective item located in inspection

Reference to Figure 12 will aid in the construction of the utility function. All defectives located in inspection are assumed to be repaired and returned to the lot.

The utility of inspecting a lot under the decision function

$$d(x) = \begin{cases} \text{accept lot, if } x \le c \\ \text{rectify lot, if } x > c \end{cases} \tag{6-4}$$

when x defectives are observed and X defectives are in the lot, is given by:

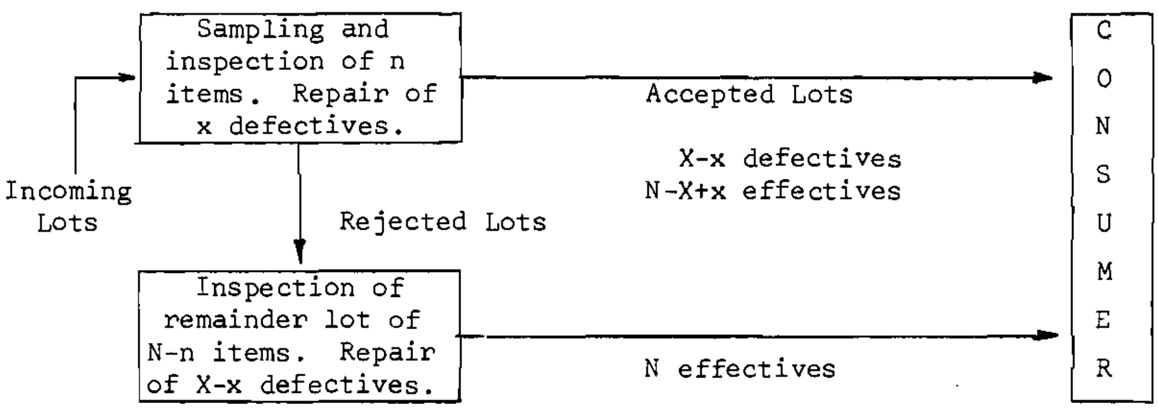


Figure 12. Flow Diagram for Inspection with Rectification.

$$u(x,n,c,X) = \begin{cases} V_1(N-X+x) + V_2(X-x) - (s_1+s_2)n - S_F\delta - C_r x - C_M N, \\ \qquad \text{if } x \le c \\ V_1 N - s_1 n - s_2 N - C_r X - S_F \delta - C_M N, \text{ if } x > c \end{cases} \tag{6-5}$$

where

$$\delta = \begin{cases} 1, \text{ if inspection is carried out} \\ 0, \text{ otherwise.} \end{cases} \tag{6-6}$$

Let $d_a$ and $d_r$ be the two decision functions which ignore all sample data. That is,

$d_a(x)$ = accept for all $x$ , and

$d_r(x)$ = reject for all $x$.

Under both of these special cases the optimal sample size is obviously zero. Therefore

$$u(x,d_a,X) = V_1(N - X) + V_2 X - C_M N \tag{6-7}$$

and

$$u(x,d_r,X) = V_1 N - s_2 N - C_r X - C_M N - S_F \tag{6-8}$$

Note that $d_a$ is the policy of no inspection and $d_r$ is the policy of detail inspection.

The utility characteristic, $\bar{u}(d,X)$, is the expected value of $u(x,d,X)$ with respect to x, given X,

$$\bar{u}(d,X) = \sum_{x=0}^{c} [V_1(N-X+x) + V_2(X-x) - (s_1+s_2)n - S_F\delta - C_rX] f(x|X) \quad (6\text{-}9)$$

$$+ \sum_{x=c+1}^{n} [V_1N-s_1n-s_2N-C_rX-S_F\delta] f(x|X) - C_MN$$

$$= V_1N-s_1n-s_2N+s_2(N-n) \sum_{x=0}^{c} f(x|X) - C_rX$$

$$- (V_1-V_2-C_r) \sum_{x=0}^{c} (X-x)f(x|X) - C_MN - S_F\delta \ .$$

If the prior distribution of X is specified and Bayes principle is to be used, Equation (6-10) is needed to compute the expected utility associated with a choice of d. (The probability distributions used are defined in Chapter III.)

$$U(d) = \sum_{X=0}^{N} \bar{u}(d,X)h(X) \quad (6\text{-}10)$$

$$= V_1N - s_1n - s_2N - C_rE(X) + s_2(N-n) \sum_{X=0}^{N} \sum_{x=0}^{c} f(x|X)h(X)$$

$$- (V_1-V_2-C_r) \sum_{X=0}^{N} \sum_{x=0}^{c} (X-x)f(x|X)h(X) - C_MN - S_F\delta$$

$$= (V_1-s_2-C_M)N - s_1n - C_rN\bar{p} + s_2(N-n) \sum_{x=0}^{c} g_n(x)$$

$$- (V_1-V_2-C_r) \sum_{x=0}^{c} \sum_{y=0}^{N-x} yf(y|x)g_n(x) - S_F\delta \qquad \text{(6-10) Continued}$$

$$= (V_1-C_M-s_2-C_r\bar{p})N - s_1n - S_F\delta + s_2(N-n)\,G_n(c)$$

$$- (V_1-V_2-C_r) \sum_{x=0}^{c} E(y|x)\; g_n(x)$$

$$= (V_1-C_M-s_2-C_r\bar{p})N - s_1n - S_F\delta + s_2(N-n)\; G_n(c)$$

$$- (V_1-V_2-C_r)\,\frac{(N-n)}{(n+1)} \sum_{x=0}^{c} (x+1)\; g_{n+1}(x+1)\ .$$

The optimal policy of the type specified by Equation (6-4) maximizes U(d). The no-data decision functions $d_a$ and $d_r$ must be considered as well as the cases where $n > 0$. The expected utility for a rule of acceptance without inspection would be

$$U(d_a) = (V_1-C_M)N - (V_1-V_2)N\,\bar{p}\ , \qquad (6\text{-}11)$$

since $E(X) = N\bar{p}$. The expected utility for a rule of rejection without sampling inspection would be

$$U(d_r) = (V_1-C_M-s_2-C_r\bar{p})N - S_F\ . \qquad (6\text{-}12)$$

A policy of no inspection is preferred to a policy of detail inspection if $U(d_a)$ exceeds $U(d_r)$. This condition implies $\bar{p}$, the process average fraction defective, is less than $p_o$, an indifference quality, given by

$$p_o = \frac{s_2 + S_F/N}{V_1 - V_2 - C_r} \quad . \tag{6-13}$$

Comparison of Equation (6-10) with Equation (6-12) reveals that detail inspection is optimal only if

$$s_1 n - s_2(N-n)\ G_n(c) + (V_1 - V_2 - C_r)\ \frac{(N-n)}{(n+1)} \sum_{x=0}^{c} (x+1)\ g_{n+1}\ (x+1) > 0 \quad . \tag{6-14}$$

<u>Binomial Prior Distribution.</u> It is interesting to explore the nature of an optimal solution when the prior distribution is binomial; that is

$$h_N(X) = \binom{N}{X} \bar{p}^X \bar{q}^{N-X} \ , \ X = 0, 1, \ldots, N \ . \tag{6-15}$$

In Chapter III, the resulting $g_n(x)$ was shown to be

$$g_n(x) = \binom{n}{x} \bar{p}^x \bar{q}^{n-x} \ , \ x = 0, 1, \ldots, n. \tag{6-16}$$

Using this result in Equation (6-10),

$$U(d) = (V_1 - C_M - s_2 - C_r\bar{p})N - s_1 n - S_F\delta + s_2(N-n)G_n(c) - (V_1 - V_2 - C_r)(N-n)\bar{p}G_n(c), \tag{6-17}$$

since

$$\frac{(N-n)}{(n+1)} \sum_{x=0}^{c} (x+1)\, g_{n+1}(x+1) = \frac{(N-n)}{(n+1)} \sum_{x=0}^{c} (x+1) \binom{n+1}{x+1} \bar{p}^{x+1} \bar{q}^{n-x} = (N-n)\bar{p}G_n(c).$$

For any n, U(d) is maximized by choosing c such that $G_n(c) = 1$, which means that the lot is accepted, or $G_n(c) = 0$, which means that the lot is rejected (and rectified). Since the decision function is independent of x, the optimal sample size is zero. This restricts the choice to $d_a$ or $d_r$. Using Equation (6-13) the optimal decision function is

$$d^* = \begin{cases} d_a, \text{ if } \bar{p} \le \dfrac{s_2 + S_F/N}{(V_1-V_2-C_r)} \\ \\ d_r, \text{ if } \bar{p} > \dfrac{s_2 + S_F/N}{(V_1-V_2-C_r)} \end{cases} \quad . \tag{6-18}$$

The conclusion from this analysis can be stated as follows: If the output from which lots are formed comes from a controlled process with average fraction defective $\bar{p}$ (binomial prior distribution), do not inspect the lots if $\bar{p} \le p_o$ and detail inspect the lots if $\bar{p} > p_o$, where $p_o$ is given by Equation (6-13).

Selection of an Attribute Single Sampling Plan for Nonrectifying Inspection

Suppose that instead of being rectified, rejected lots are scrapped. Further assume that any defectives found in samples of accepted lots also are scrapped. Let $V_3$ be the value of a scrapped unit. The flow diagram for this system is shown in Figure 13.

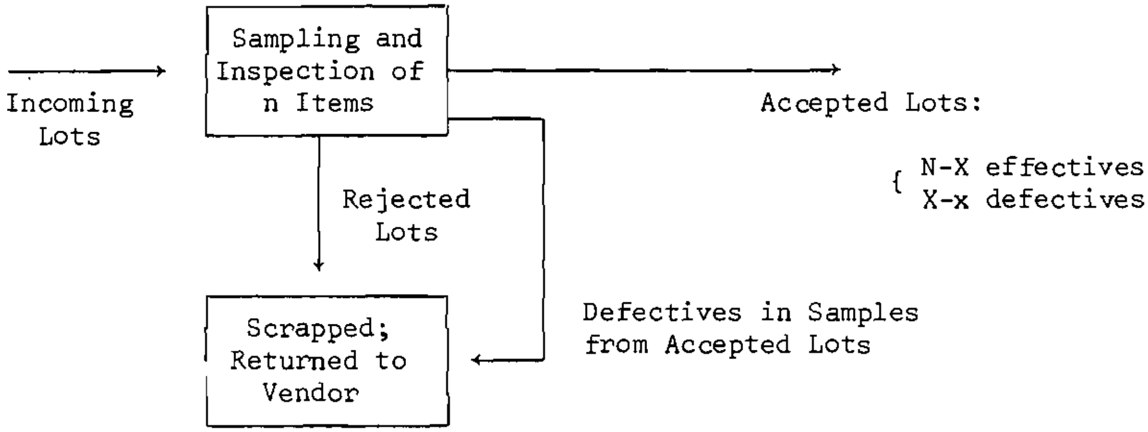


Figure 13. Flow Diagram for Inspection Without Rectification.

The utility of the decision function

$$d(x) = \begin{cases} \text{accept lot, if } x \le c. \\ \text{reject lot, if } x > c. \end{cases} \tag{6-19}$$

when x defectives are observed and X defectives are in the lot is given by

$$u(x,d,X) = \begin{cases} V_1(N-X)+V_2(X-x)-C_MN-(s_1+s_2)n-S_F\delta+V_3x, \text{ if } x \le c \\ V_3N-C_MN-(s_1+s_2)n-S_F\delta, \text{ if } x > c \end{cases} \tag{6-20}$$

where

$$\delta = \begin{cases} 0, & n = 0 \\ 1, & n > 0 \end{cases} \tag{6-21}$$

The utility characteristic is

$$\bar{u}(d,X) = \sum_{x=0}^{c} [V_1(N-X)+V_2(X-x)-C_M N-(s_1+s_2)n-S_F\delta+V_3X]f(x|X) \quad (6\text{-}22)$$

$$+ \sum_{x=c+1}^{n} [V_3N-C_M N-(s_1+s_2)n-S_F\delta]f(x|X)$$

$$= V_3N-C_M N-(s_1+s_2)n-S_F\delta+[(V_1-V_3)N-(V_1-V_2)X] \sum_{x=0}^{c} f(x|X)$$

$$+ (V_3-V_2) \sum_{x=0}^{c} x\, f(x|X) \ .$$

Using the prior distribution of X, the expected utility associated with a choice d is given by

$$U(d) = \sum_{X=0}^{N} \bar{u}(d,X)h(X) \quad (6\text{-}23)$$

$$= V_3N-C_M N-(s_1+s_2)n-S_F\delta+(V_1-V_3)N \sum_{X=0}^{N} \sum_{x=0}^{c} f(x|X)h(X)$$

$$- (V_1-V_2) \sum_{X=0}^{N} X \sum_{x=0}^{c} f(x|X)h(X)+(V_3-V_2) \sum_{X=0}^{N} \sum_{x=0}^{c} xf(x|X)h(X)$$

$$= (V_3-C_M)N-(s_1+s_2)n-S_F\delta+(V_1-V_3)NG_n(c)$$

$$- (V_1-V_2)[\sum_{x=0}^{c} x\, g_n(x) + \sum_{x=0}^{c} E(y|x)\, g_n(x)]$$

$$+ (V_3-V_2) \sum_{x=0}^{c} x\, g_n(x)$$

$$= (V_3-C_M)N - (s_1+s_2)n - S_F\delta + (V_1-V_3)N\ G_n(c) \qquad (6\text{-}23)$$

Continued

$$- (V_1-V_3) \sum_{x=0}^{c} x\ g_n(s) - (V_1-V_2) \frac{(N-n)}{(n+1)} \sum_{x=0}^{c} (x+1)\ g_{n+1}(x+1) \ .$$

The optimal decision policy of the type specified by Equation (6-19) maximizes U(d). The no-data decision functions $d_a$ and $d_r$ must be considered as well as cases where $n > 0$. The expected utility for a rule of acceptance without inspection would be

$$U(d_a) = (V_1-C_M)N - (V_1-V_2)N\bar{p} \ , \qquad (6\text{-}24)$$

while the expected utility for a rule of rejection without inspection would be

$$U(d_r) = (V_3-C_M)N \ . \qquad (6\text{-}25)$$

Acceptance without inspection is preferred to rejection without inspection if $U(d_a)$ exceeds $U(d_r)$. This condition implies that

$$\bar{p} < \frac{V_1-V_3}{V_1-V_2} \ . \qquad (6\text{-}26)$$

Comparison of Equations (6-23) and (6-25) reveals that rejection without inspection is desirable only if

$$(s_1+s_2)n + S_F\delta - (V_1-V_3)N\ G_n(c) + (V_1-V_3) \sum_{x=0}^{c} x\ g_n(x) \qquad (6\text{-}27)$$

$$- (V_1-V_2)\,\frac{(N-n)}{(n+1)} \sum_{x=0}^{c} (x+1)\; g_{n+1}\,(x+1) > 0 \;. \qquad (6\text{-}27)$$

Continued

Binomial Prior Distribution. If the prior distribution is given by Equation (6-15),

$$U(d) = (V_3-C_M)N - (s_1+s_2)n - S_F\delta + (V_1-V_3)G_n(c) - (V_1-V_3)n\bar{p}\; G_{n-1}(c) \qquad (6\text{-}28)$$

$$- (V_1-V_2)(N-n)\bar{p}\; G_n(c) \;,$$

since

$$\sum_{x=0}^{c} x\; g_n(x) = n\bar{p} \sum_{x=0}^{c} g_{n-1}(x) = n\bar{p}\; G_{n-1}(c) \;.$$

For any n, U(d) is maximized by choosing c such that

$$\begin{cases} G_n(c) = 1, \text{ if } \bar{p} \le \dfrac{(V_1-V_3)N}{(V_1-V_2)N + (V_2-V_3)n} \\[2ex] G_n(c) = 0, \text{ if } \bar{p} > \dfrac{(V_1-V_3)N}{(V_1-V_2)N + (V_2-V_3)n} \end{cases} \qquad (6\text{-}29)$$

Therefore for a chosen n, the decision function is independent of x:

$$d(x) = \begin{cases} \text{Accept, if } \bar{p} \le \dfrac{(V_1-V_3)N}{(V_1-V_2)N + (V_2-V_3)n} \\[2ex] \text{Reject, if } \bar{p} > \dfrac{(V_1-V_3)N}{(V_1-V_2)N + (V_2-V_3)n} \;. \end{cases} \qquad (6\text{-}30)$$

Since for any n, use of (6-28) yields

$$U(d) = \begin{cases} (V_3-C_M)N - (s_1+s_2)n - S_F\delta + (V_1-V_3)N - (V_1-V_3)n\bar{p} - (V_1-V_2)(N-n)\bar{p}, \\ \qquad \text{if } \bar{p} \le \dfrac{(V_1-V_3)N}{(V_1-V_2)N + (V_2-V_3)n} \\ (V_3-C_M)N - (s_1+s_2)n - S_F\delta, \text{ if } \bar{p} > \dfrac{(V_1-V_3)N}{(V_1-V_2)N + (V_2-V_3)n} \end{cases} \tag{6-31}$$

the optimal choice for n is zero. The optimal decision rule is therefore

$$\begin{cases} d_a = \text{Accept without inspection if } \bar{p} \le \dfrac{(V_1-V_3)}{(V_1-V_2)} \\ d_r = \text{Reject without inspection if } \bar{p} > \dfrac{(V_1-V_3)}{(V_1-V_2)} . \end{cases} \tag{6-32}$$

The binomial prior distribution was analyzed to show that sampling inspection of the output of a process in statistical control is not economic. The lot should be either accepted or rejected on the basis of the process average fraction defective and its relation to an indifference quality level. This fact results from the zero correlation between the number of defectives in the sample and the quality of the remainder lot. For other prior distributions (Polya, mixed binomial) there will be conditions under which sampling inspection is the most economic policy. This is shown by Hald (53) and Suzuki (109).

## Discussion of the Utility Functions

The utility functions of the previous two examples are not presented as being appropriate for all situations. However, they include many possibly relevant factors and are more complex than most economic models appearing in the literature of acceptance sampling. Also, these measures of effectiveness are gain to the decision maker rather than the universally used concept of losses.

The various revenues and costs were assumed to be linear functions of the sample size, lot size, or number of defectives. Functions other than linear could be used if they more accurately represented the behavior of the economic measure as the independent variable changed.

The inspection costs were represented by three parameters: the cost of sampling a unit ($s_1$), the cost of inspecting a unit ($s_2$), and a fixed cost per lot ($S_F$) if sampling inspection is undertaken. It was implicit in the formulation of the model that all items sampled were inspected and that the unit cost of detail inspection is the same as that of sampling inspection. The fixed cost component represents the cost of handling the lot, inventory carrying costs because of delay in processing, and other costs, fixed with respect to the decision function, put on a per lot basis.

The parameter $V_1$ represents the value associated with an accepted effective item. This could be taken as the sale price of the item less the remaining costs of manufacture and selling, or in the case of a component part or purchased material, $V_1$ could be taken as the difference in profit between passing a defective and passing an effective, if $V_2$ is set equal to zero.

These parameters are difficult to evaluate. In general they cannot be obtained from accounting records. Their estimation is part of the engineering research of an inspection operation.[1]

## Summary

In this chapter a conceptual model of an inspection operation and a system of interrelated operations was presented. The applicability of statistical decision theory as a method for selection of inspection procedures was demonstrated. The approach to system optimization through the application of statistical decision theory was illustrated for a single-stage system. Two examples were given to illustrate the methodology: the first was an analysis of rectifying inspection and the second was an analysis of non-rectifying inspection. For a binomial prior distribution, it was shown that sampling inspection is never optimal. This indicates that an important savings in inspection cost is possible for a manufacturer who controls his process fraction defective at a level below the indifference quality $p_o$.

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1. Raiffa and Schlaifer (88) suggest that an unknown cost parameter be treated as a future and a subjective probability distribution assigned to it. If T is the set of all possible values of the cost parameter t, the compound set of futures would be $\{\theta,t\}$, having prior distribution $h(\theta,t)$. The Bayes analysis would require taking expectation over t as well as $\theta$. For a utility function linear in t, this amounts to replacing t with $E(t)$.

# CHAPTER VII

# ECONOMIC SELECTION OF ACCEPTANCE CRITERIA FOR ITEM INSPECTION FOR DEFECTS IN A MULTISTAGE PRODUCTION PROCESS

## General

In this chapter, the problem of choosing acceptance criteria for defects inspection is analyzed to evaluate the economic feasibility of severe or tightened specification limits at in-process inspection operations. This type of problem was selected for several reasons. It illustrates the analysis of a multistage process; the solution can be obtained through use of dynamic programming, a technique that seems to hold promise for analysis of multistage inspection operations; it was concerned with detail inspection, as opposed to the usual emphasis on acceptance sampling; it was motivated by actual industrial inspection problems, which indicates that the solution is of practical value; and it is a type of problem which apparently has not been considered in the literature.

Two environmental situations are considered. In the first case, a manufacturer is producing to meet a fixed goal, so that any defective items must be replaced by reprocessing a substitute. In the second situation, a manufacturer is producing from a fixed stock of raw material, so that a rejected item results in a loss in revenue.

A method of treating the problem when defects are not of equal importance is discussed.

## Production to a Fixed Quota

### Statement of the Problem

Units are processed through a series of M production operations, in each of which defects may be acquired. Assume that the number of defects acquired at operation k are independent of those acquired at any other operation and that no defects are removed by subsequent operations. A unit is to be inspected before each operation, and at these points the cumulative number of defects may be noted. Suppose the maximum number of defects to be tolerated is $D_M$. Units with defects in excess of $D_M$ are considered defective and are assumed to have no value. Naturally, if at any inspection the cumulative number of defects exceeds $D_M$, the unit is removed; however, it may be desirable to consider tightening the specifications (i.e., rejecting a unit with less than $D_M$ defects) at the earlier stages of production. Thus, at the inspection after operation k, the specification would be of the following form:

> Reject the unit if $T_k > D_k$, where $T_k$ is the cumulative number of defects through operation k and $D_k$ is a non-negative integer satisfying $D_k \le D_M$. Otherwise accept the unit for processing in stage k + 1.

The problem is to determine the decision rule at each stage, given acquisition costs, the distribution of the number of defects produced at each stage, the value of a completed good unit, and the maximum allowable number of defects for a completed unit.

To make explicit the cost of rejection of a nondefective item, assume that items removed at any stage k when $D_k < T_k \le D_M$ are set aside and carefully processed to completion under conditions controlled so that no more defects are introduced. Assume further that this special treatment results in increased production costs.

Assuming that a fixed production quota has been established, any defective items must be replaced. This means acquiring and starting a new unit. Assume that all makeup of production shortages is done under the above-mentioned controlled conditions, so that no defects are generated.

Symbolic Formulation of the Problem

The following notation is introduced to allow construction of a symbolic model of this decision problem:

$d_k$ = number of defects introduced by production operation k ($k = 1, 2, \ldots, M$).

$d_o$ = number of defects in an item prior to production operation 1.

$T_k = d_o + d_1 + \ldots + d_k$, for $k = 0, 1, 2, \ldots, M$ .

$C_o$ = value of an item before processing (acquisition cost).

$C_k$ = cost of processing a unit at stage k, ($k = 1, 2, \ldots, M$), which includes cost of inspection after production operation k.

$S_k$ = cost of special processing of a unit from stage k through stage M ($S_o$ includes $C_o$).

V = value associated with a unit having completed stage M with $T_M \leq D_M$ .

$R_k(T_{k-1},D_{k-1})$ = gain at stage k when $T_{k-1}$ defects are present after stage k-1 and the specification $D_{k-1}$ is used in inspection before stage k.

$f_{M-k}(T_k)$ = maximum expected gain using optimal policies in the last M-k stages, when $T_k$ defects are present after k stages.

This notation gives rise to the recurrence relation (for $k = 0, 1, \ldots, M-1$),

$$f_{M-k}(T_k) = \max_{D_k} \mathop{E}_{d_{k+1}} \{R_{k+1}(T_k,D_k) + f_{M-k-1}(T_{k+1})\} \tag{7-1}$$

where

$$f_o(T_M) = \begin{cases} V, \text{ if } T_M \le D_M \\ V-S_o, \text{ if } T_M > D_M \end{cases} \tag{7-2}$$

and[1]

$$R_{k+1}(T_k,D_k) = \begin{cases} -C_{k+1}, & T_k \le D_k \\ V-S_{k+1}-f_{M-k-1}(T_{k+1}), & D_k < T_k \le D_M \\ V-S_o-f_{M-k-1}(T_{k+1}), & D_M < T_k\ . \end{cases} \tag{7-3}$$

The problem is solved by successively finding $D^*_{M-1}$, $D^*_{M-2}$, ..., $D^*_o$, values

1. Since $V-S_{k+1}-f_{M-k-1}(T_{k+1}) > V-S_o-f_{M-k-1}(T_{k+1})$ (because $S_{k+1} < S_o$ by assumption), $D_k$ will not exceed $D_M$ .

which accomplish the above indicated maximizations. The expected gain associated with an unprocessed unit having $d_o$ defects is then $f_M(d_o)$ when the optimal set of specifications is used. The technique is illustrated in the following example.

Analysis of a Three-Stage Process

The item is processed through three production stages, which generate defects according to a Poisson process having means 3, 2, and 6, respectively. The following data are available:

| | | |
|---|---|---|
| $C_o = 40$ | $S_o = 130$ | $V = 100$ |
| $C_1 = 10$ | $S_1 = 90$ | $D_3 = 16$ |
| $C_2 = 20$ | $S_2 = 70$ | |
| $C_3 = 15$ | $S_3 = 30$ | |

By definition,

$$f_o(T_3) = \begin{cases} V, \text{ if } T_3 \le D_3 \\ V - S_o, \text{ if } T_3 > D_3 . \end{cases}$$

Considering the decision at the inspection before stage 3 and referring to (7-1), one finds that $D_2$ should be chosen to maximize

$$\underset{d_3}{E} \{R_3(T_2,D_2) + f_o(T_3)\} .$$

Since

$$R_3(T_2,D_2) = \begin{cases} -C_3, \text{ if } T_2 \le D_2 \\ V - S_3 - f_o(T_3), \text{ if } D_2 < T_2 \le D_3 \\ V - S_o - f_o(T_3), \text{ if } D_3 < T_2 \end{cases}$$

and $T_3 = T_2 + d_3$,

$$\underset{d_3}{E}\{R_3(T_2,D_2) + f_o(T_3)\} = \begin{cases} \underset{d_3}{E}\,[-C_3 + f_o(T_2 + d_3)], \text{ if } T_2 \le D_2 \\ V - S_3, \text{ if } D_2 < T_2 \le D_3 \\ V - S_o, \text{ if } D_3 < T_2 \,. \end{cases}$$

Since $D_2 \le D_3$,

$$\underset{d_3}{E}\{R_3(T_2,D_2) + f_o(T_3)\} = \begin{cases} V \sum_{d_3=0}^{D_3-T_2} P_3(d_3) + (V-S_o) \sum_{d_3=D_3-T_2+1}^{\infty} P_3(d_3) - C_3, \\ \qquad \text{if } T_2 \le D_2 \\ V - S_3, \text{ if } D_2 < T_2 \le D_3 \\ V - S_o, \text{ if } D_3 < T_2 \,. \end{cases}$$

where $P_3(d_3)$ is the probability distribution of $d_3$. Examination of the above expression reveals that $D_2^*$ should be selected as the largest value of $T_2$ for which

$$V\,F(D_3-T_2) + (V-S_o)[1-F(D_3-T_2)] - C_3 \ge V - S_3$$

or

$$F_3(D_3-T_2) \geq \frac{S_o + C_3 - S_3}{S_o}$$

where $F_3(\cdot)$ is the distribution function of $P_3(\cdot)$.[2] (Note that the choice of $D_2^*$ is independent of V.) For the data given above, it is required that

$$F_3(D_3-T_2) \geq \frac{115}{130} = 0.885 \ .$$

Examination of a cumulative Poisson table reveals $D_3-T_2 = 9$ is the smallest value for which $F_3(D_3-T_2) \geq 0.885$. Therefore we choose $D_2^* = 7$. For this value of $D_2$

$$f_1(T_2) = \begin{cases} V - S_o\,[1 - F_3(D_3-T_2)] - C_3, \text{ if } T_2 \leq D_2^* \\ V - S_3, \text{ if } D_2^* < T_2 \leq D_3 \\ V - S_o, \text{ if } D_3 < T_2 \ . \end{cases}$$

Values of this function are given in Table 7.

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2. Since the relations $0 \leq C_3 \leq S_3 \leq S_o$ are implicit in the definitions of $S_3$, $C_3$, and $S_o$, it follows that

$$0 \leq \frac{S_o + C_3 - S_3}{S_o} \leq 1 \ .$$

Table 7. Values of the Function $f_1(T_2)$

| $T_2$ | $f_1(T_2)$ | $T_2$ | $f_1(T_2)$ |
|---|---|---|---|
| 0 | 85.0 | 9 | 70.0 |
| 1 | 84.9 | 10 | 70.0 |
| 2 | 84.9 | 11 | 70.0 |
| 3 | 84.5 | 12 | 70.0 |
| 4 | 83.8 | 13 | 70.0 |
| 5 | 82.4 | 14 | 70.0 |
| 6 | 79.4 | 15 | 70.0 |
| 7 | 74.1 | 16 | 70.0 |
| 8 | 70.0 | ≥17 | -30.0 |

Next the optimal two-stage policy must be determined. $D_1^*$ is chosen to maximize

$$\underset{d_2}{E}\ \{R_2(T_1,D_1) + f_1(T_2)\} \ .$$

By Equation (7-3):

$$R_2(T_1,D_1) = \begin{cases} -\ C_2, \text{ if } T_1 \leq D_1 \\ V - S_2 - f_1(T_2), \text{ if } D_1 < T_1 \leq D_3 \\ V - S_o - f_1(T_2), \text{ if } D_3 < T_1 \end{cases}$$

and therefore

$$\underset{d_2}{E}\{R_2(T_1,D_1) + f_1(T_2)\} = \begin{cases} \underset{d_2}{E}\{-C_2 + f_1(T_1+d_2)\}, & T_1 \le D_1 \\ V - S_2, & D_1 < T_1 \le D_3 \\ V - S_o, & D_3 < T_1 \end{cases}$$

$$= \begin{cases} \sum_{d_2=0}^{D_3-T_1} f_1(T_1+d_2)P_2(d_2) + (V-S_o)[1-F_2(D_3-T_1)] - C_2, & \text{if } T_1 \le D_1 \\ V - S_2, & D_1 < T_1 \le D_3 \\ V - S_o, & D_3 < T_1 \,. \end{cases}$$

The optimal value for $D_1$ is the largest value of $T_1$ for which

$$\sum_{d_2=0}^{D_3-T_1} f_1(T_1+d_2) + (V-S_o)[1-F_2(D_3-T_1)] - C_2 \ge V-S_2 \,.$$

For the given data, $D_1^*$ was found to be 13. The expected gain using the optimal two-stage policy is

$$f_2(T_1) = \begin{cases} \sum_{d_2=0}^{D_3-T_1} f_1(T_1+d_2)P_2(d_2) + (V-S_o)[1-F_2(D_3-T_1)] - C_2, & \text{if } T_1 \le D_1^* \\ V - S_2, & \text{if } D_1^* < T_1 \le D_3 \\ V - S_o, & \text{if } D_3 < T_1 \,. \end{cases}$$

Values of this function are given in Table 8.

Finally, the optimal three-stage strategy is determined by choosing $D_o^*$ to maximize

$$\mathop{E}_{d_1} \{R_1(T_o,D_o) + f_2(T_1)\} .$$

Table 8. Values of the Function $f_2(T_1)$

| $T_1$ | $f_2(T_1)$ | $T_1$ | $f_2(T_1)$ |
|---|---|---|---|
| 0 | 64.53 | 9 | 49.96 |
| 1 | 63.97 | 10 | 49.50 |
| 2 | 62.91 | 11 | 48.30 |
| 3 | 61.12 | 12 | 44.70 |
| 4 | 58.51 | 13 | 34.70 |
| 5 | 55.33 | 14 | 30.00 |
| 6 | 52.38 | 15 | 30.00 |
| 7 | 50.55 | 16 | 30.00 |
| 8 | 50.00 | ≥17 | -30.00 |

As before,

$$R_1(T_o,D_o) = \begin{cases} - C_1, \text{ if } T_o \le D_o \\ V - S_1 - f_2(T_1), \text{ if } D_o < T_o \le D_3 \\ V - S_o - f_2(T_1), \text{ if } D_3 < T_o \end{cases}$$

and

$$\underset{d_1}{E}\ \{R_1(T_o,D_o) + f_2(T_1)\}$$

$$= \begin{cases} \sum_{d_1=0}^{D_3-T_o} f_2(T_o+d_1)P_1(d_1) + (V-S_o)[1-F_1(D_3-T_o)] - C_1, & T_o \le D_o \\ V - S_1, & D_o < T_o \le D_3 \\ V - S_o, & D_3 < T_o\ . \end{cases}$$

$D_0^*$ is the largest value of $T_o$ such that

$$\sum_{d_1=0}^{D_3-T_o} f_2(T_o+d_1)P_1(d_1) + (V-S_o)[1-F_1(D_3-T_o)] - C_1 > V - S_1 \quad .$$

For the data of this example, the optimal value of $D_o$ is 12. The expected gain using the optimal three-stage policy is

$$f_3(T_o) = \begin{cases} \sum_{d_1=0}^{D_3-T_o} f_2(T_o+d_1)P_1(d_1) + (V-S_o)[1-F_1(D_3-T_o)] - C_1, & \text{if } T_o \le D_0^* \\ V - S_1, & D_0^* < T_o \le D_3 \\ V - S_o, & D_3 < T_o\ . \end{cases}$$

Values of this function are given in Table 9.

Table 9. Values of the Function $f_3(T_o)$

| $T_o$ | $f_3(T_o)$ | $T_o$ | $f_3(T_o)$ |
|---|---|---|---|
| 0 | 50.29 | 9 | 31.37 |
| 1 | 48.18 | 10 | 26.01 |
| 2 | 45.84 | 11 | 19.12 |
| 3 | 43.55 | 12 | 10.61 |
| 4 | 41.70 | 13 | 10.00 |
| 5 | 40.28 | 14 | 10.00 |
| 6 | 39.06 | 15 | 10.00 |
| 7 | 47.58 | 16 | 10.00 |
| 8 | 35.15 | ≥17 | -30.00 |

The optimal inspection policy may be summarized as follows:

If an item has acquired no more than 12 defects prior to operation one, process it through operation one; if it then has 13 or less defects process it through operation two; and if it then has 7 or less defects, process it through stage three. Set aside any items which fail to meet these specifications and process them as a special run.

Comparison with a System Having Only Final Inspection

If there are no in-process inspections for defects, the expected gain from processing an item presented to the first production stage will be

$$(V - \sum_{i=1}^{M} C_i)\ P\{T_M \le D_M\} + (V - S_o - \sum_{i=1}^{M} C_i)\ P\{T_M > D_M\}\ . \tag{7-4}$$

This may be compared with the gain using an optimal policy for in-process inspection, averaged over all values of $d_o$, i.e.,

$$\sum_{d_o=0}^{\infty} f_M(d_o)\ P(d_o)\ . \tag{7-5}$$

For the example given in the previous section, suppose that $d_o = 0$ for all items. Then in Equation (7-4) $T_M$ is Poisson with mean 11, resulting in an expected gain per item of

$$(100-45)\ P(T_M \leq 16) + (100-130-45)\ P(T_M > 16)$$

$$= (55)(.944) + (-75)(.056) = 47.72 \ .$$

The value for in-process inspection is $f_3(0) = 50.29$. In-process inspection for defects does seem justified in case all items arrive free of defects at the first operation.

Now suppose that $d_o$ has a Poisson distribution with mean 3. Then equation (7-4) yields 23.28, while Equation (7-5) gives 44.19. The difference in gain of 20.91 would justify in-process inspection for defects.

## Production from a Fixed Stock

### Problem Statement

To illustrate the analysis under a different set of assumptions, the following problem is considered. A manufacturer is processing a certain style of product from a fixed supply of raw material which he already owns. As before, the units are processed through M consecutive production operations with inspection for defects preceding and following each operation. A completed item having a total number of defects, $T_M$, no more than a specified quantity $D_M$ is considered "first quality" and is sold for a revenue (net of packing, transportation, and selling costs) of V per unit. Completed units with $D_M < T_M \leq D_M'$ are graded "second quality" and are sold at a discount for a net revenue of V' per unit.

The problem is to determine optimal rules for deciding whether a unit, which has acquired $T_k$ defects through operation k, should be processed further or should be removed from the process. Assume that units removed before completion of processing or units which are completed but have $T_M > D_M'$ are sold as scrap for an average net revenue of L per unit. It seems logical to require $L < V' < V$ .

Formulation of the Model

The recurrence relation given in Equation (7-1) holds here, but Equations (7-2) and (7-3) must be modified to conform to the new assumptions. The new forms of $f_o(T_M)$ and $R_{k+1}(T_k,D_k)$ are given below:

$$f_o(T_M) = \begin{cases} V \text{ , if } T_M \le D_M \\ V'\text{, if } D_M < T_M \le D_M' \\ L \text{ , if } D_M' < T_M \end{cases} \tag{7-6}$$

$$R_{k+1}(T_k,D_k) = \begin{cases} -C_k\text{, if } T_k \le D_k \\ L - f_{M-k-1}(T_{k+1})\text{, if } T_k > D_k \end{cases} \tag{7-7}$$

The procedure for determining the optimal solution is the same as illustrated previously in this chapter; however, it is instructive to consider the determination of $D_{M-1}^*$.

Determination of $D_{M-1}^*$

Referring to Equation (7-1) and using Equations (7-6) and (7-7), one finds that $D^*_{M-1}$ is the value of $D_{M-1}$ which maximizes

$$\underset{d_M}{E}\{R_M(T_{M-1},D_{M-1}) + f_o(T_M)\} = \begin{cases} \underset{d_M}{E}\{-C_M + f_o(T_{M-1}+d_M)\}, \text{ if } T_{M-1} \leq D_{M-1} \\ L, \text{ if } T_{M-1} > D_{M-1} \end{cases}$$

$$= \begin{cases} VF_M(D_M-T_{M-1}) + V'[F_M(D_M'-T_{M-1}) - F_M(D_M-T_{M-1})] + L[1-F_M(D_M'-T_{M-1})] - C_M, \\ \qquad \text{if } T_{M-1} \leq D_{M-1} \\ L, \text{ if } T_{M-1} > D_{M-1} \end{cases} \tag{7-8}$$

Therefore, $D^*_{M-1}$ is the largest value of $T_{M-1}$ for which

$$VF_M(D_M-T_{M-1}) + V'[F_M(D_M'-T_{M-1}) - F_M(D_M-T_{M-1})] + L[1-F_M(D_M'-T_{M-1})] - C_M > L$$

or, after some simplification,

$$(V-V')F_M(D_M-T_{M-1}) + (V'-L)F_M(D_M'-T_{M-1}) > C_M \ . \tag{7-9}$$

Once $D^*_{M-1}$ is found, $f_1(T_{M-1})$ is known and can be used to find $D^*_{M-2}$, and so on, until the optimal policy has been determined.

## Correction of Defects at Subsequent Operations

The economic principles are unchanged if a production operation normally removes some of the defects acquired by an item in prior operations; however, a more complicated probability structure is involved.

The variable $d_k$ is now defined as the change in the number of defects when the item is processed in production operation k. Its probability distribution would be conditional on $T_{k-1}$. A large amount of information about the process would be required in order to estimate these distributions. To compute the optimal policy, the iterative procedure defined by Equation (7-1) still could be used, provided expectations are taken with respect to the proper conditional distributions.

## Use of a Weighted Defect Total

It may be that all defects are not equally important and a specification based on the total number of defects (of all types) is not an accurate measure of the quality of an item. One solution is to establish separate specifications for each type of defect and declare defective any item which fails to conform to any specification. Another approach is to assign a weight function w to the class of defect types, such that a weighted defect total may be computed at each inspection. If $x_{jk}$ is the number of defects of type j found in inspection k, the weight defect total is given by

$$T_k = \sum_j w_j X_{jk} . \tag{7-10}$$

$T_k$ then plays the role it did in the earlier analysis; only now it may be interpreted as the cumulative number of "demerits" that the item has acquired. A single specification is established for $T_M$.

The choice of the specification $D_M$ is somewhat subjective and is related to the requirements of the user of the product. If the user is able to state the maximum allowable number of defects of each type (say, $D_{M1}$, $D_{M2}$, ..., $D_{Mj}$, ...), then $D_M$ could be got from

$$D_M = \sum_j w_j D_{Mj} . \tag{7-11}$$

## Summary

When an item has to be processed through a sequence of production operations and there is a maximum number of defects allowed a completed unit, there can exist some economic advantage in tightening the specifications for in-process inspections. This hypothesis was examined with respect to two problem environments: (1) a manufacturer producing to satisfy a fixed production quota and (2) a manufacturer producing until he exhausts a fixed stock of raw material. Under specific assumptions regarding the disposition of rejected product, a dynamic programming model was formulated for each problem. A numerical example was solved for a three-stage process under the first set of assumptions. The results of these analyses indicate artificially severe specification limits should be considered and that dynamic programming methods can be utilized to determine the most economic limits.

The use of a weighted defect total was suggested when various types of defects exist and they are not all of the same degree of severity. The specification limits would be established for the weighted total.

# CHAPTER VIII

# CONCLUSIONS AND RECOMMENDATIONS

## Conclusions

The results and conclusions evolving from this research are summarized in the following paragraphs.

A study of the literature of acceptance inspection resulted in these conclusions regarding present practice and research:

1. Efforts to utilize economic criteria and knowledge of the process curve in the development of inspection procedures have not resulted in any generally applicable theory.

2. There is no general agreement on the proper measure of effectiveness for evaluating acceptance inspection procedures.

3. There is no general agreement on the appropriate principle of choice for selecting among alternative inspection procedures.

4. Inspection has not been treated adequately as a system of interrelated operations.

5. Types of inspection schemes presently available provide adequate choice for the form of an acceptance inspection decision rule.

6. Improvements in the design of acceptance inspection systems will result from a better developed and more clearly understood set of principles--economic, statistical, mathematical, psychological--than now exists relative to acceptance inspection.

The application of statistical decision theory to inspection problems revealed a need for development of both statistical and economic principles and methods. To satisfy the statistical requirements, expressions were developed for the following quantities:

1. Conditional probability distribution of the sample outcome, given the lot quality.

2. Joint probability distribution of the sample outcome and the lot quality.

3. Marginal distribution of the sample outcome.

4. Conditional distribution of lot quality, given the sample outcome.

5. Probability of accepting a lot of a given quality.

6. Expected quality reaching the consumer under rectifying inspection.

7. Expected quality reaching the consumer under nonrectifying inspection.

8. Average size of lots reaching the consumer.

9. Expected number of units inspected per lot.

These results were obtained for single sampling plans for attribute inspection for defectives, attribute inspection for defects, and variables inspection with known process standard deviation. Specific attention was given the following prior distributions: hypergeometric, binomial, Polya, and mixed binomial for defectives inspection; gamma and m-point discrete distributions for defects inspection; and normal distribution for variables inspection.

The three categories of losses affected by inspection decisions are inspection costs, acceptance losses, and rejection losses. Monetary units represent the best measure of effectiveness for inspection decisions. The estimation of the relevant losses is of about the same order of difficulty as in most engineering economy studies.

It is concluded that the Bayes principle, possibly constrained by specification of aspiration levels, is the most logical principle of choice for acceptance inspection decisions.

Results were obtained which quantify the effect of inspection errors by inspectors and instruments. A procedure was developed to determine the economically optimal number of replicate measurements to make when inspecting an item.

The methodology of statistical decision theory was shown to be applicable to the analysis of inspection operations. The use of decision theory in practice is made difficult by lack of knowledge of the prior distribution, uncertainty about estimates of losses, and complexity of computations required to obtain a solution. However, the theory does properly describe the decision problem, thereby giving the analyst an understanding of the nature of the problem which he is attempting to solve. In those cases where the prior distribution can be described adequately and losses can be estimated accurately, use of the theory permits choice of a system which best satisfies the objectives of the decision maker. Using this approach it was demonstrated that acceptance sampling is not optimal when the prior distribution is binomial and the loss function is linear.

The applicability of the decision theory approach to multistage inspection systems was demonstrated for defects inspection. It is concluded that artificially severe rejection criteria may be economic for in-process inspection.

## Recommendations

In the course of this investigation and as a result of the structuring of the principles and methods for inspection system design, several potentially useful areas of research were revealed. These areas are discussed in the remainder of this section.

There is a need for the development of procedures for estimating the various gains and losses which are influenced by inspection decisions. This may involve the design of special study procedures or the modification of accounting methods to permit the accumulation of relevant data on a routine basis.

A similar need exists with regard to the estimation of the prior distribution of lot quality. Techniques for gathering and analyzing data in order to determine the pattern of variation in lot quality would be of value in the application of decision theory to acceptance inspection.

To correctly assess the effect that a given acceptance inspection system will have on the quality of material produced by a process, a better understanding of the psychological influence exerted upon workers because of the presence and nature of inspection is required. Research which generates additional knowledge about these phenomena would be useful.

Because system characteristics (such as costs, prior distributions, specifications) change over time, a procedure for adjusting the inspection system is required. This would involve the acquisition and analysis of data to determine whether or not the existing inspection system should be modified. Efforts could be devoted to developing general methods for accomplishing this control function.

Additional work on the application of decision theory to acceptance inspection would prove valuable. In particular, the development of results for lot-by-lot inspection procedures for sampling designs other than simple random sampling and the extension of economic analysis of sequential schemes are two areas deserving more research.

To give further aid in the application of these principles, three additional areas of research are indicated: determination of the optimal design of single-stage inspection systems for particular prior distributions and particular utility functions; development and application of multistage optimization techniques to the design of acceptance inspection systems; and actual application to a variety of processes, with detailed documentation of problems encountered and results obtained.

Many of the results of this research were based upon the assumption of a stationary prior distribution. It would be useful to investigate similar problems when this assumption is relaxed.

Finally, it should be restated that in this research only inspection for the purpose of determining the disposition of the product was considered. The benefits of using inspection information for other purposes, such as process control, were not analyzed. In order to optimize the design of the entire quality control system, it will be necessary to consider a multitude of factors, including particularly the two primary reasons for inspection: determination of product acceptability and process control. The importance of this overall problem requires that more research be undertaken.

# APPENDICES

# APPENDIX A

## PARTIAL CLASSIFICATION OF THE LITERATURE RELEVANT TO THE ECONOMIC ANALYSIS OF ACCEPTANCE INSPECTION

The numbers under the headings refer to articles in the bibliography.

### Type of Acceptance Inspection

#### Detail Inspection

73, 79, 96.

#### Attribute Single Sampling

16, 17, 18, 42, 52, 61, 63, 77, 85, 86, 88, 102, 103, 107, 109, 116, 118, 120, 123, 127.

#### Variables Single Sampling

28, 35, 51.

#### Attribute Sequential Sampling

5, 6, 17, 18, 21, 22, 72, 86, 102, 119, 120.

#### Variables Sequential Sampling

72, 128.

#### Continuous Inspection

4, 26, 30, 31, 41, 47, 66, 67, 93, 94.

### Principles of Choice

#### Bayes

4, 6, 9, 10, 16, 21, 26, 28, 42, 52, 53, 63, 69, 72, 93, 94, 102, 103, 107, 109, 118, 119, 127, 128.

Minimax (Loss or Regret)

5, 17, 18, 85, 107, 116, 120.

Aspiration Level

14, 20, 33, 69, 72, 110.

Minimax (Game Theory)

63, 120.

Disposition of Rejected Lots

Rectifying Inspection of Remainder Lots

5, 28, 33, 53, 63, 85, 86, 90, 95, 102, 103, 107, 109, 120, 123, 130.

Refusal to Accept Rejected Lots

7, 17, 18, 42, 69, 95, 120.

Prior Distribution

Mixed Binomial

6, 16, 61, 53, 84, 103, 118, 119, 127.

Polya

53, 84, 86, 88.

Beta

21, 53, 88, 102, 103, 107, 109, 110, 118.

Poisson

5.

Sensitivity Analysis

Prior Distribution

6, 86, 103, 118.

Costs

103.

General Discussions of Economics of Acceptance Inspection

2, 3, 16, 44, 59, 60, 62, 73, 75, 76, 82, 103, 111, 123.

Preventive Aspects of Inspection

60, 64, 74, 129.

Multistage Systems

9, 10, 63, 77, 79, 96.

Inspection Errors

19, 24, 70, 89.

Quality of Design

1, 36, 45, 104.

Statistical Decision Theory

6, 12, 61, 81, 88, 95, 122.

Tables or Charts for Choosing Inspection Procedures

Noneconomic

14, 20, 33, 43, 105, 112, 113, 114, 115.

Economic

17, 18, 68.

Miscellaneous Economic Analyses

13, 83, 87, 97.

Standard Texts on Quality Control Procedures

14, 15, 25, 33, 34, 39, 44, 62, 73, 99, 100, 125.

Sampling Techniques Other Than Simple Random Sampling

23, 29.

# APPENDIX B

## VARIANCE OF THE CONDITIONAL DISTRIBUTION, $f(y|x)$

Hald (53, p. 295) gives the rth factorial moment of $f(y|x)$ as

$$\mu_{(r)}(x) = (N-n)^{(r)} \ (x+r)^{(r)} \ g_{n+r}(x+r) \ / \ [(n+r)^{(r)} \ g_n(x)] \ , \qquad \text{(B-1)}$$

where

$$x^{(r)} = x(x-1)\ (x-2)\ \cdots\ (x-r+1)\ .$$

To find the second ordinary moment about the mean, the relation

$$\mu_2 = \mu_{(2)} - \mu_1^2 + \mu_1 \qquad \text{(B-2)}$$

is used.

The mean is $\mu_{(1)}$:

$$\mu_{(1)}(x) = (N-n) \left[\frac{(x+1)\ g_{n+1}(x+1)}{(n+1)\ g_n(x)}\right] . \qquad \text{(3-18)}$$

The second factorial moment is

$$\mu_{(2)}(x) = \frac{(N-n)(N-n-1)(x+2)(x+1)\ g_{n+2}(x+2)}{(n+2)(n+1)g_n(x)} . \qquad \text{(B-3)}$$

Given $g_n(x)$ and the observed value for x, Equations (3-18), (B-2), and (B-3) can be used to calculate $V(y|x)$:

$$V(y|x) = \left[\frac{(N-n)(N-n-1)(x+2)(x+1)g_{n+2}(x+2)}{(n+2)(n+1)g_n(x)}\right] - E(y|x)[E(y|x)-1] \ . \quad (3\text{-}19)$$

# APPENDIX C

## DERIVATION OF CERTAIN RESULTS

## FOR ATTRIBUTE INSPECTION FOR DEFECTS

1. Calculation of the Mean and Variance of $g_n(x)$, Given by Equation (3-78).

Derivation of the result given by Equation (3-79):

$$E(x) = \sum_{x=0}^{\infty} x \sum_{\lambda} f_n(x|\lambda)h(\lambda) = \sum_{\lambda} h(\lambda) \sum_{x} x\ f_n(x|\lambda) = n \sum_{\lambda} \lambda\ h(\lambda) = n\bar{\lambda} \quad .$$

Derivation of the result given by Equation (3-80):

$$V(x) = E(x^2) - [E(x)]^2$$

$$= [\sum_{x} x^2 \sum_{\lambda} f_n(x|\lambda)\ h(\lambda)] - (n\bar{\lambda})^2$$

$$\overset{*}{=} \sum_{\lambda} h(\lambda)[(n\lambda)^2 + n\lambda] - (n\bar{\lambda})^2$$

$$= n^2[E(\lambda^2) - \bar{\lambda}^2] + n\bar{\lambda}$$

$$= n^2 V(\lambda) + n\bar{\lambda} \quad .$$

$$* \sum_{x=0}^{\infty} x^2\ f_n(x|\lambda) = \sum_{x=0}^{\infty} x(x-1)\ f_n(x|\lambda) + \sum_{x=0}^{\infty} x\ f_n(x|\lambda) = (n\lambda)^2 + (n\lambda) \quad .$$

2. Calculation of the Variance of $g_n(x)$, Given by Equation (3-97).

Equation (3-98) is derived as follows: Using Equation (3-80),

$$V(x) = n^2[\sum_{i=1}^{m} w_i\lambda_i^2-\bar{\lambda}^2] + n \sum_{i=1}^{m} w_i\lambda_i$$

$$= n^2 \sum_{i=1}^{m} w_i\lambda_i^2 - n^2\bar{\lambda}^2 + n\bar{\lambda}$$

$$= n^2 \sum_{i=1}^{m} w_i\lambda_i^2 - n\bar{\lambda}(n\bar{\lambda}-1) .$$

3. Derivation of the Conditional Distribution, $f(\lambda|x)$.

Equations (3-99) and (3-100) are derived as follows:

$$f(\lambda|x) = \frac{f(x,y)}{g_n(x)}$$

$$= \frac{w_i e^{-n\lambda_i} (n\lambda_i)^x}{\sum_{i=1}^{m} w_i e^{-n\lambda_i}(n\lambda_i)^x} \equiv w_i(x) .$$

4. Derivation of the Mean and Variance of $f(\lambda|x)$, Given by Equation (3-99).

Equation (3-101) is derived as follows:

$$E(\lambda|x) = \sum_{i=1}^{m} \lambda_i W_i(x) = \sum_{i=1}^{m} \lambda_i \frac{w_i e^{-n\lambda_i}(n\lambda_i)^x}{\sum_{i=1}^{m} w_i e^{-n\lambda_i}(n\lambda_i)^x}$$

$$= \frac{(x+1)}{n} \frac{\sum_{i=1}^{m} w_i \; e^{-n\lambda_i}(n\lambda_i)^{x+1}/(x+1)!}{\sum_{i=1}^{m} w_i \; e^{-n\lambda_i} \; (n\lambda_i)^x/x!}$$

$$= \frac{(x+1) \; g_n(x+1)}{n \; g_n(x)} \; .$$

Equation (3-102) is derived as follows:

$$V(\lambda|x) = \sum_{i=1}^{m} \lambda_i^2 \; w_i(x) - [E(\lambda|x)]^2$$

$$= \sum_{i=1}^{m} \lambda_i^2 \frac{w_i \; e^{-n\lambda_i} \; (n\lambda_i)^x}{\sum_{i=1}^{m} w_i \; e^{-n\lambda_i} \; (n\lambda_i)^x} - [E(\lambda|x)]^2$$

$$= \frac{(x+2)(x+1)}{n^2} \; \frac{g_n(x+2)}{g_n(x)} - \left[\frac{(x+1) \; g_n(x+1)}{n \; g_n(x)}\right]^2$$

$$= \frac{(x+1)}{n^2 g_n^2(x)} [(x+2) \; g_n(x+2) \; g_n(x) - (x+1) \; g_n(x+1)] \quad .$$

5. <u>Calculation of the Mean of $E(y|x)$, Given by Equation (3-106).</u>

Equation (3-107) is derived as follows:

$$E(y|x) = \sum_{y=0}^{\infty} y \; f(y|x) = \sum_{y=0}^{\infty} y \sum_{\lambda} f(\lambda|x) \; e^{-(N-n)\lambda} [N-n)\lambda]^y/y!$$

$$= \sum_{\lambda} f(\lambda|x) \sum_{y=0}^{\infty} y\ e^{-(N-n)\lambda}[(N-n)\lambda]^{y}/y!$$

$$= \sum_{\lambda} [(N-n)\lambda]\ f\ (\lambda|x) = (N-n)\ \lambda_{x}\ .$$

# APPENDIX D

## DERIVATION OF THE MEAN AND VARIANCE OF $g_n(m;\sigma)$

1. The mean of $g_n(m;\sigma)$ is given by Equation (3-117). It is derived below:

$$E(m) = \int_{-\infty}^{\infty} m \int_{-\infty}^{\infty} h(\mu)\ f_n(m|\mu)\ d\mu\ dm$$

$$= \int_{-\infty}^{\infty} h(\mu) \int_{-\infty}^{\infty} m\ f_n(m|\mu)\ dm\ d\mu$$

$$= \int_{-\infty}^{\infty} \mu\ h(\mu)\ d\mu = \bar{\mu} \quad .$$

2. The variance of $g_n(m;\sigma)$ is given by Equation (3-118). It is derived below:

$$V(m) = E(m^2) - [E(m)]^2$$

$$= \int_{-\infty}^{\infty} m^2 \int_{-\infty}^{\infty} h(\mu)\ \ f_n(m|\mu)\ d\mu\ dm - \bar{\mu}^2$$

$$= \int_{\infty}^{\infty} h(\mu) \int_{\infty}^{\infty} m^2\ \ f_n(m|\mu)\ dm\ d\mu - \bar{\mu}^2$$

$$= \int_{\infty}^{\infty} (\sigma^2/n + \mu^2)\ h(\mu)\ d\mu - \bar{\mu}^2$$

$$= \sigma^2/n + E(\mu^2) - \bar{\mu}^2$$

$$= \sigma^2/n + V(\mu) + \bar{\mu}^2 - \bar{\mu}^2 = \sigma^2/n + V(\mu) \quad .$$

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# VITA

Lynwood Albert Johnson was born October 4, 1933, at Macon, Georgia, the son of Benjamin Albert and Dorothy Harriet (neé Mallard) Johnson. He attended public schools in Macon, graduating first in a class of approximately 300 from Lanier High School.

He entered the Georgia Institute of Technology in September, 1951. During his four years at Georgia Tech, he was elected to Tau Beta Pi, Phi Kappa Phi, Phi Eta Sigma, Alpha Pi Mu, Koseme, the Rambling Reck Club, the Arnold Air Society, and Who's Who in American Colleges and Universities. Mr. Johnson was named the outstanding sophomore student in the School of Industrial Engineering in 1953 and the outstanding senior student in Industrial Engineering in 1955. He was named the outstanding AFROTC cadet in his class during each of the three years he was in the program. He was a three-year member of the varsity golf team. In addition, he is a member of the Kappa Alpha Order social fraternity and was president of the Georgia Tech chapter during the 1954-55 academic year. He graduated in June, 1955, with the degree Bachelor of Industrial Engineering with Highest Honor.

From August, 1955 to September, 1957, Mr. Johnson worked as an industrial engineer in the Methods and Standard Department at the Savannah River Plant of E. I. DuPont de Nemours & Company. On November 3, 1956, he was married to Beaufort Sims Law.

Mr. Johnson returned to Georgia Tech in the fall of 1957 to begin graduate studies in the School of Industrial Engineering. He was awarded the Callaway Fellowship for the 1957-58 academic year. Work on his program for the degree Master of Science in Industrial Engineering was completed in August, 1958. A thesis entitled *A Control Model for Waste Performance in a Cotton Spinning Mill* was written by him under the direction of Dr. David C. Elkey.

He joined the faculty of the School of Industrial Engineering as an Instructor in September, 1958. At the same time he was enrolled in the Ph.D. program in the field of industrial engineering with emphasis on operations research methodology. He was promoted to Assistant Professor in 1960. His teaching assignments were in the areas of engineering economy, engineering statistics, operations research, production and inventory control, and quality control at the undergraduate level and inventory control, engineering economy, and experimental statistics at the graduate level.

During the summer of 1961, he studied in the Statistics Department at Iowa State University, under the sponsorship of the National Science Foundation and the Georgia Tech Foundation.

In addition to teaching duties, he was active in the adult education program of Georgia Tech and served as a part-time member of the staff of the Rich Electronic Computer Center from July, 1962 to June, 1964. He is co-author of a text, *Introduction to Linear Programming*, to be published by Prentice-Hall, Inc. He is a member of the American Institute of Industrial Engineers, the Operations Research Society, and the Institute of Management Sciences. He is a registered professional engineer in Georgia.

In July, 1964, Mr. Johnson joined Kurt Salmon Associates, Inc., as Senior Associate in their Operations Research Division.